ARBITRARY SMALL INDIVISIBILITIES[◇]

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Abstract

Arbitrary small indivisibilities may play an important role when the strong survival assumption does not hold. A hierarchic price is a finite ordered family of price vectors $\{p_1, \ldots, p_k\}$. It extends the notion of exchange values proposed by Gay [15]. These price notions were introduced in order to establish the existence of a generalized competitive equilibrium without the strong survival assumption.

We show that a hierarchic price models phenomena related to small indivisibilities which the standard approach may not capture. More precisely, we prove in the framework of linear exchange economies that a hierarchic price may be seen as a standard price of an economy with arbitrary small indivisibilities.

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1. Introduction

The perfect divisibility of commodities is one of the crucial assumptions in general equilibrium theory. Perfect divisibility of goods allows for applying fixed point theorems establishing existence of equilibria. This corresponds of course to an idealized representation of the commodity space. The rational is that the commodities one considers are "almost perfectly" divisible in the sense that the indivisibilities are small and insignificant enough so that they can be neglected. So the question arises when indivisibilities may be insignificant. Is it sufficient that they are small or is more needed?

In the absence of indivisibilities, at any strictly positive price, it is possible to exchange a unit of good A against a positive quantity of some good B and thus all consumers have access to all commodities provided their income is not zero. If commodities are not perfectly divisible, it may be impossible to exchange a unit of good A against good B, provided good B is expensive enough. So consumers may not all have access to the same commodities, i.e. the consumer who initially has no good B may have no access to this commodity, if the minimal unit of B is worth more than his initial endowment. This may occur at any level of indivisibility of the commodities.

This problem may not arise if the strong survival assumption holds, i.e. if every consumer has his initial endowment in the interior of his consumption set. Then, every consumer has initially a positive quantity of every commodity and therefore he can consume a positive quantity of every commodity, no matter what the price is. However, this assumption is highly unrealistic since most consumers have a single commodity to sell - their labour. Without such an assumption a Walras equilibrium may fail to exist, even when goods are perfectly divisible. Several authors (Gay [15], Danilov and Sotskov [6], Marakulin [18], Florig [9]) proposed generalized competitive equilibrium concepts existing without a strong survival assumption.

These equilibrium notions differ from the standard approach in the price notion they use. In fact, by considering a standard price, i.e. a linear function on the commodity space, and working with a convex subset of $\mathbb{R}^\ell$ as consumption set, it may be impossible to capture phenomena related to indivisibilities as alluded above. This can be done working with a hiearchic price. A hierarchic price is an ordered family of price vectors of the form $\{p_1, \ldots, p_k\}$[1]. Each price vector corresponds to some "submarket". With

[1] Note that hierarchic prices have also been used in the value and in the market mechanism literature by Mertens [20],[21].

commodities of submarket two it is not possible to buy commodities of submarket one, but commodities of submarket three are almost free with respect to a unit of some commodity of submarket two. In Florig [9], a hierarchic price was interpreted as the idealized representation of some price vector of the form $p(\varepsilon) = p_1 + \varepsilon p_2 + \ldots + \varepsilon^{k-1} p_k$ for some small $\varepsilon > 0$. The level of $\varepsilon > 0$ should depend on the level of indivisibility of the commodities and vanish when the level of indivisibility converges to zero. If commodities are divisible into an arbitrarily large number of units, then the hierarchic price would be in some sense the limit point of the price vector $p(\varepsilon)$ with ε converging to zero. So a hierarchic equilibrium (Marakulin [18], Florig [9]) should be seen as a competitive equilibrium capable to capture problems related to indivisibilities, provided they are very small.

With a discrete consumption set, for a given price, it may be possible that some consumer wants to consume a bundle which costs a little bit less than the value of his initial endowment. So similar problems as with satiation points occur (cf. Drèze and Müller [7], Makarov [17], Aumann and Drèze [1], Mas Colell [19]). In order to attain an equilibrium it may be necessary to relax the budget constraint of some consumers slightly leading to a dividend equilibrium (Aumann and Drèze [1]) also called a competitive equilibrium with slack (Mas Colell [19]). As Kahji [16] pointed out the slack or dividends in the consumers' budget constraint may be interpreted as paper money. So a dividend equilibrium may be seen as a Walras equilibrium with money and a possibly positive value of money.

To give a formal proof of the validity of our interpretation of a hierarchic equilibrium one should be able to construct for any hierarchic equilibrium a sequence of dividend equilibria of economies without perfectly divisible goods. When the level of indivisibility converges to zero, the hierarchic equilibrium should then be the limit point of the sequence of dividend equilibria. However, Shapley and Scarf [22] gave an example of an economy without a divisible commodity and an empty core and a forciori without a Walras or dividend equilibrium. Their example satisfies standard assumptions apart the perfect divisibility of commodities. For this reason we will focus here on a particular class of economies with a very rich structure, i.e. linear exchange economies. In a linear exchange economy, consumers preferences are supposed to be representable by linear functions. Such economies have been extensively studied in Gale [12],[13],[14], Eaves [8], Cornet [5], Mertens [21], Bonnisseau, Florig and Jofré [3],[4] and Florig [10]. Although we have no existence result for dividend equilibria in linear exchange economies without perfectly divisible economies at hand, their rich structure will enable us to construct for each hierarchic equilibrium a sequence of dividend equilibria converging to it as indivisibilities go to zero. For this we will however suppose that all consumers have rational initial endowments and utility functions represented by a rational vector.

This result gives a formal argument confirming the kind of interpretation proposed for hierarchic prices in Florig [9] - at least for a certain class of economies. Thus in the absence of the survival assumptions even small indivisibilities may play a crucial role.

Since a hierarchic equilibrium reduces to a Walras equilibrium under standard sufficient conditions for the existence of a Walras equilibrium, we may deduce as a corollary that every Walras equilibrium can be obtained as a limit point of dividend equilibria of economies with vanishing indivisibilities. So in this case, indivisibilities are indeed negligible provided they are small.

In Section 2, we will state the model and the result, in section 3 we give some concluding remarks and in section 4, we proof our result.

2. Model and Result

We consider a linear exchange economy with a finite set $L = \{1, \ldots, \ell\}$ of commodities and $I = \{1, \ldots, m\}$ of consumers. Every consumer is characterized by a consumption set X_i, with $\mathbb{Z}^\ell_+ \subset X_i \subset \mathbb{R}^\ell_+$, an initial endowment $\omega_i \in \mathbb{R}^\ell_+$ and a utility function $u_i : \mathbb{R}^\ell_+ \to \mathbb{R}$ defined by $u_i(x_i) = b_i \cdot x_i$ for some given vector $b_i \in \mathbb{R}^\ell_+$. For convenience we will assume that $\sum_{i \in I} \omega_i \in \mathbb{R}^\ell_{++}$. A linear exchange economy is a family

$$\mathcal{L}(X_i, b_i, \omega_i)_{i \in I}.$$

Let $\overline{\mathbb{R}} = (\mathbb{R} \cup \{+\infty\})$. For all $n \in \mathbb{N}$, let $\geq_{lex}$ be the lexicographic order on $\overline{\mathbb{R}}^n$ [2]. For $x \in \mathbb{R}^n$, let $\text{supp}(x) = \{h \in \{1, \ldots, n\} \mid x^h \neq 0\}$ be the support of x.

Definition 2.1. (i) A *hierarchic price* is an ordered family $\mathcal{P} = \{p_1, \ldots, p_k\}$ of vectors of $\mathbb{R}^\ell$.

(ii) An *exchange value* is an ordered family $\mathcal{P} = \{p_1, \ldots, p_k\}$ of vectors of $\mathbb{R}^\ell_+ \setminus \{0\}$ such that $(\text{supp}(p_1), \ldots, \text{supp}(p_k))$ is a partition of L.

[2] For $(s, t) \in (\overline{\mathbb{R}}^n)^2$, $s \geq_{lex} t$, if $s_r < t_r$, $r \in \{1, \ldots, n\}$ implies that $\exists \rho \in \{1, \ldots, r-1\}$ such that $s_\rho > t_\rho$. We write $s >_{lex} t$ if $s \geq_{lex} t$, but not $[t \geq_{lex} s]$.

We note the set of hierarchic prices by $\mathcal{HP}$ and the set of exchange values by $\mathcal{EV}$. We will see that for linear exchange economies with $X_i = I\!R^\ell_+$ for all $i \in I$, there is no loss of generality in considering only exchange values. The number k is determined at equilibrium, if $k = 1$ we are back in the standard case.

For $\mathcal{P} \in \mathcal{HP}$ and $x \in I\!R^\ell$, we define the *value* of x to be

$$\mathcal{P}x = (p_1 \cdot x, \ldots, p_k \cdot x) \in \overline{I\!R}^k.$$

We come now to the budget sets of the consumers. A *hierarchic revenue* will be an element $w \in \overline{I\!R}^k$. For $i \in I$, $\mathcal{P} \in \mathcal{HP}$, $w \in \overline{I\!R}^k$ and $X_i \subset I\!R^\ell$ let

$$r_i(X_i, \mathcal{P}, w) = \min\{r \in \{1, \ldots, k\} \mid \exists x \in X_i, (p_1 \cdot x, \ldots, p_r \cdot x) <_{lex} (w_1, \ldots, w_r)\},$$

with $r_i(X_i, \mathcal{P}, w) = k$, if the minimum is taken over the empty set,

$$v_i(X_i, \mathcal{P}, w) = (w_1, \ldots, w_{r_i(X_i, \mathcal{P}, w)}, +\infty, \ldots, +\infty) \in \overline{I\!R}^k.$$

The *budget set* of consumer $i \in I$ with respect to the hierarchic price $\mathcal{P} \in \mathcal{HP}$, to the hierarchic revenue w and the consumption set X_i is

$$B_i(X_i, \mathcal{P}, w) = \{x_i \in X_i \mid \mathcal{P}x_i \leq_{lex} v_i(X_i, \mathcal{P}, w)\}.$$

The *demand* of consumer $i \in I$ with respect to the hierarchic price $\mathcal{P} \in \mathcal{HP}$, to the hierarchic revenue w and the consumption set X_i is

$$D_i(X_i, \mathcal{P}, w) = \begin{cases} \text{argmax } u_i(x_i) = b_i \cdot x_i \\ \mathcal{P}x_i \leq_{lex} v_i(X_i, \mathcal{P}, w) \\ x_i \in X_i \end{cases}$$

Definition 2.2. (a) A *hierarchic equilibrium* of the economy $\mathcal{L}(X_i, b_i, \omega_i)_{i \in I}$ is a collection $((x_i, w_i)_{i \in I}, \mathcal{P}) \in \prod_{i \in I}(X_i \times \overline{I\!R}^k) \times \mathcal{HP}$ such that :

(i) for all $i \in I$, $x_i \in D_i(X_i, \mathcal{P}, w_i)$;

(ii) for all $i \in I$, $\mathcal{P}\omega_i \leq_{lex} w_i$;

(iii) $\Sigma_{i \in I} x_i = \Sigma_{i \in I} \omega_i$.

(b) A *dividend equilibrium* of the economy $\mathcal{L}(X_i, b_i, \omega_i)_{i \in I}$ is a hierarchic equilibrium $((x_i, w_i)_{i \in I}, p) \in \prod_{i \in I}(X_i \times I\!R) \times I\!R^\ell$.

(c) A *Walras equilibrium* of the economy $\mathcal{L}(X_i, b_i, \omega_i)_{i\in I}$ is a dividend equilibrium $((x_i, w_i)_{i\in I}, p) \in \prod_{i\in I}(X_i \times I\!R) \times I\!R^\ell$ such that for all $i \in I$, $w_i = p \cdot \omega_i$.

The extra revenue $w_i - \mathcal{P}\omega_i$ may be seen as the value of an initial holding of some paper money which does not enter consumers' preferences (cf. Kahji [16], Florig [9]). Note that if for all $i \in I$, $r_i(X_i, \mathcal{P}, w_i) = 1$, i.e. for example if a strong survival assumption holds, then a hierarchic equilibrium is a dividend equilibrium. If moreover non-satiation of the preferences holds, then it is a Walras equilibrium. The converse is of course always true. The hierarchic equilibrium extends the generalized equilibrium notions proposed in Gay [15] and Danilov and Sotskov [6]. It is defined in Florig [9] in a general framework including production. Its existence is proven in Florig [9] under standard conditions replacing the strong survival assumption ($\omega_i \in \mathrm{int}X_i$ for all i) by a weak one, i.e. $\omega_i \in X_i$ for all i. Using non-standard analysis, Marakulin [18] proves the existence of a generalized equilibrium concept for exchange economies. In an exchange economy with polyhedral consumption sets (hence in the present framework), a hierarchic equilibrium and Marakulin's [18] generalized equilibrium coincide.

We will consider economies where all commodities are indivisible. For every $n \in I\!N$, let us denote

$$X^n = \{x \in I\!R^\ell_+ \mid \exists \mu \in Z\!\!Z^\ell_+, \forall h \in L, x_h = \mu_h/n\}.$$

Note that for any strictly increasing mapping $\phi : I\!N \to I\!N$, $\lim_{n\to+\infty} X^{\phi(n)} = I\!R^\ell_+$, in the sense of Painlevé-Kuratowski.

Example 1. First let us give a trivial example. There are two agents 1,2 and two goods 1,2. Let $X_1 = X_2 = I\!R^2_+$, $\omega_1 = (1,1), \omega_2 = (0,1)$ and the utility functions are $u_1(x) = x^1$ and $u_2(x) = x^1 + x^2$. Neither a Walras nor a dividend equilibrium exist. The hierarchic equilibria are $\mathcal{P} = \{p_1, p_2\}$ with $p_1 = (1,0), p_2 = (0,1)$, $x_1 = (1,t), x_2 = (0, 2-t)$, $w_1 = (1, +\infty), w_2 = (0, 2-t)$, $t \in [0,1]$. Suppose we may divide any commodity into n minimal units. Then the following is a dividend equilibrium of the corresponding discrete economy: $p^n = (1, \frac{1}{3n})$, $x_1^n = (1, t^n), x_2^n = (0, 2-t^n)$, $w_1^n = 1 + \frac{1}{3n}, w_2^n = \frac{1}{3n}(2-t^n)$ with $t^n \in [t, \min\{1, t+1/n\}]$ such that there exists $\mu \in Z\!\!Z_+$ satisfying $t^n = \mu/n$.

So whatever the level of indivisibility, consumer 2 cannot access the market of good 1 at equilibrium. The hierarchic price enables us to approximate a discrete consumption space by $I\!R^2_+$ capturing the phenomena observed at any level of indivisibility, that is: no one has enough of good 2 in order to use it for buying some of good 1; the price of good 2 is far from being small for consumer 2 but almost zero for consumer 1.

Example 2. Consider an exchange economy with three consumers and three commodities; for all $i \in I$, $X_i = I\!R^3_+$, $u_1(x) = x^1$, $u_2(x) = x^1 + 2x^2 + x^3$, $u_3(x) = x^1 + x^2 + 2x^3$, $\omega_1 = (1,1,1), \omega_2 = \omega_3 = (0,0,0)$. The hierarchic equilibrium allocations are $x_1 = (1,b,c), x_2 = (0,1-b,t), x_3 = (0,0,1-c-t)$ for $b,c \in [0,1]$, $t \in [0,1-c]$, with the hierarchic price $p_1 = (1,0,0), p_2 = (0,2,1)$ and $y_1 = (1,b,c), y_2 = (0,1-b-t,0), y_3 = (0,t,1-c)$ for $b,c \in [0,1]$, $t \in [0,1-b]$ with the hierarchic price $q_1 = (1,0,0), q_2 = (0,1,2)$. It is sufficient to take $p^n = p_1 + \frac{1}{4n}p_2$ and $q^n = q_1 + \frac{1}{4n}q_2$ in order to approach the respective hierarchic equilibria by dividend equilibria of the economy where each commodity may divide any commodity into n minimal units.

The limit of both price sequences is $p = (1,0,0)$. At this price one could exchange good 2 against good 3 at any exchange rate. So p is not a good approximation of the equilibrium price of a discrete economy. We would totally neglect at which rate one may exchange good 2 against good 3 and again goods 2 and 3 are far from being cheap for consumers 2 and 3.

Example 3. The following example does not fit in the framework of our theorem below. Neither a Walras nor a dividend equilibrium nor a hierarchic equilibrium with an exchange value exists. Here one may not interpret the hierarchic equilibrium price in terms of sub-markets. The interpretation in terms of indivisibilities of the hierarchic price remains nevertheless valid here. There are two agents 1,2 and two goods 1,2. Let $X_1 = X_2 = \{x \in I\!R^2_+ \mid x^1 + x^2 \geq 3\}$, $\omega_1 = (3,1)$, $\omega_2 = (2,1)$ and the utility functions are $u_1(x) = x^1$ and $u_2(x) = x^2$. It is easy to check that the only hierarchic equilibrium is $\mathcal{P} = \{p_1, p_2\}$ with $p_1 = (1,1), p_2 = (-1,1)$, $x_1 = (4,0), x_2 = (1,2)$, $w_1 = (4,-2), w_2 = (3,1)$. For any discretization of the consumption set (as described above) containing x_1 and x_2, there exists a real $\varepsilon > 0$ such that $(p_1 + \varepsilon p_2, x_1, x_2)$ is a dividend equilibrium. Taking a sequence of discretizations converging to the initial convex consumption set one obtains at the limit the hierarchic equilibrium.

The following result shows formally that the interpretation we attributed to hierarchic prices are valid not only for the preceding numerical examples, but more generally for linear exchange economies, provided the utility functions and initial endowments are represented by rational vectors.

Theorem 2.3. *Suppose for all* $i \in I$, $(b_i, \omega_i) \in Q^\ell_+ \times Q^\ell_+$. *Let* $((x_i, w_i)_{i\in I}, \mathcal{P})$ *with* $\mathcal{P} = \{p_1, \dots, p_k\} \in \mathcal{EV}$ *be a hierarchic equilibrium of* $\mathcal{L}(I\!R^\ell_+, b_i, \omega_i)_{i\in I}$.

Then, there exist strictly increasing functions $\phi : I\!N \to I\!N$, $\varphi : I\!N \to I\!N$ together with a sequence $((x_i^n)_{i\in I}, \mathcal{P}^n)$ with $x_i^n \in X^{\phi(n)}$ for all $i \in I$ and $\mathcal{P}^n = \{p_1^n, \ldots, p_k^n\} \in \mathcal{EV}$ such that:

(i) $(x_i^n)_{i\in I}$ converges to $(x_i)_{i\in I}$ and for all $r \in \{1, \ldots, k\}$, p_r^n converges to p_r;

(ii) for every $n \in I\!N$, $((x_i^n, w_i^n)_{i\in I}, p^n)$ is a dividend equilibrium of $\mathcal{L}(X^{\phi(n)}, b_i, \omega_i)_{i\in I}$ where

$$p^n = \sum_{r=1}^{k} \varepsilon_r^n p_r^n \text{ with } \varepsilon_r^n = \frac{(1/\varphi(n))^{r-1}}{\| \sum_{r=1}^{k} (1/\varphi(n))^{r-1} p_r^n \|} \text{ and } w_i^n = \max\{p^n \cdot x_i^n, p^n \cdot \omega_i\};$$

(iii) there exists $\mathcal{Q} = \{q_1, \ldots, q_{k'}\} \in \mathcal{HP}$, $(w_i')_{i\in I} \in (\overline{I\!R}^{k'})^m$ and k' sequences α_r^n such that for all $n \in I\!N$, $p^n = \sum_{r=1}^{k'} \alpha_r^n q_r$, with $\lim_{n\to+\infty} \frac{\alpha_{r+1}^n}{\alpha_r^n} = 0$ for all $r \in \{1, \ldots, k'-1\}$ and such that $((x_i, w_i')_{i\in I}, \mathcal{Q}) \in (I\!R_+^\ell)^m \times \mathcal{HP}$ is a hierarchic equilibrium of $\mathcal{L}(I\!R_+^\ell, b_i, \omega_i)_{i\in I}$.

Remark. Since Walras and dividend equilibria of an economy with convex consumption sets are special cases of a hierarchic equilibrium (with $k = 1$), in the present set-up, they are of course also limits of dividend equilibria of an discretized version of the economy, when indivisibilities go to zero.

The following proposition indicates that there is no loss of generality in considering only exchange values. It is a corollary of Proposition 1 in Florig [9].

Proposition 2.4. *Let $((x_i, w_i)_{i\in I}, \mathcal{P})$ be a hierarchic equilibrium of $\mathcal{L}(I\!R_+^\ell, b_i, \omega_i)_{i\in I}$. Then, there exists an exchange value $\mathcal{Q} \in \mathcal{EV}$ and $(w_i') \in (\overline{I\!R}^k)^m$ such that $((x_i, w_i')_{i\in I}, \mathcal{Q})$ is also a hierarchic equilibrium.*

3. Conclusion

The literature on indivisible commodities usually focuses on "large and significant" indivisibilities (see Bobzin [2] for a survey). The present paper shows that in the absence of the survival assumption now matter how "small" indivisibilities are, they may play an important role. Indeed, no matter how well commodities are divisible, consumers may not all have access to the same commodities at the resulting competitive equilibrium. This phenomenon occurs of course only within an equilibrium analysis. Outside an equilibrium analysis one could be tempted to state the following misleading argument:

For a given price, if goods can be sufficiently well divided, everybody has access to all markets. So restricted access to markets occurs only with “large” indivisibilities.

In the present approach the price is only determined once the level of indivisibility is chosen and therefore restricted access to the market is possible at any level of indivisibility. This may of course not occur under a strong survival assumption. In this case all consumers have an income of the same order in the sense that they have all access to all commodities. So in the absence of the strong survival assumption the fact that no commodity is really perfectly divisible may not be neglected. However, instead of considering a discrete consumption set which is mathematically hard to work with, one may capture the same economic phenomena by considering a more complex notion of prices instead: hierarchic prices.

Finally, we did not explore the converse problem here, i.e. we did not proof whether a sequence of dividend equilibria converges in some sense to a hierarchic equilibrium as indivisibilities vanish. This could quite easily be done following the arguments in Florig [9]. Establishing such a result in a very general framework would however be of little interest since the existence of such a sequence is not ensured (cf. Shapley and Scarf [22]). A generalization of the dividend equilibrium existing when all goods are indivisible is needed first. Florig and Rivera (2001) proposed and proved existence of such an equilibrium concept for economies with a continuum of trader. They prove furthermore that such an equilibrium converges to a hierarchic equilibrium when the level of inidvisibility converges to zero.

4. Appendix

Most results on linear exchange economies established in Gale [14], Bonnisseau, Florig and Jofré [3],[4] and Florig [10], are given for economies satisfying the following conditions:

(I) $(\sum_{i=1}^m b_i, \sum_{i=1}^m \omega_i) \in I\!R^\ell_{++} \times I\!R^\ell_{++}$;

(II) for every i, $b_i \neq 0$ and $\omega_i \neq 0$,

(III) the economy has no super self sufficient subset, i.e. there exists no proper subset I' of I such that:

(i) for all $h \in L$, $\sum_{i\in I'} b_i^h > 0$ implies $\sum_{i\in I\setminus I'} \omega_i^h = 0$;

(ii) there exists $h \in L$ such that $\sum_{i\in I'} \omega_i^h > 0$, but $\sum_{i\in I'} b_i^h = 0$.

We will denote by $\mathcal{W}$ the set of economies $\mathcal{L}(\mathbb{R}^\ell_+, b_i, \omega_i)_{i\in I}$ satisfying these conditions (I)-(III). Given an economy $\mathcal{L}(X_i, b_i, \omega_i)_{i\in I}$ let $X(b_i, \omega_i)_{i\in I}$ (resp. $P(b_i, \omega_i)_{i\in I}$) be the set of Walras equilibrium allocations (resp. prices).

The following lemma will be used in the proof of Theorem 2.3.

Lemma 4.1. *Let $\mathcal{L}(\mathbb{R}^\ell_+, b_i, \omega_i)_{i\in I}$ such that $(b_i, \omega_i)_{i\in I} \in (\mathbb{Q}^\ell)^m \times (\mathbb{Q}^\ell)^m \cap \mathcal{W}$, then there exists $\{x_1, \ldots, x_\nu\} \subset (\mathbb{Q}^\ell)^m$ such that*

$$X(b_i, \omega_i)_{i\in I} = \mathrm{co}\{x_1, \ldots, x_\nu\}.$$

Proof of Lemma 4.1. By Gale (cf. Eaves [8]) there exists $p \in P(b_i, \omega_i)_{i\in I} \cap \mathbb{Q}^\ell$ (this may also be deduced from Bonnisseau, Florig and Jofré [4], section 3). By Corollary 5.2 of Florig [10] there exists $\{x_1, \ldots, x_\nu\} \subset X(b_i, \omega_i)_{i\in I}$ such that $\mathrm{co}\{x_1, \ldots, x_\nu\} = X(b_i, \omega_i)_{i\in I}$ and for every $s \in \{1, \ldots, \nu\}$, x_s is of minimal support in $X(b_i, \omega_i)_{i\in I}$. Fix some $x \in \{x_1, \ldots, x_\nu\}$ and let

$$G = \{(i, h) \in I \times L | x_i^h > 0\}.$$

Note that G may be viewed as a bipartite graph with vertices $I \cup L$ and there exists an edge between $i \in I$ and $h \in L$ if and only if $(i, h) \in G$. Note also that the graph G has no cycle, since otherwise one could construct some $\xi \in X(b_i, \omega_i)_{i\in I}$ with its support strictly included in that of x by transfers, similarly as in the proof of Proposition 4.2 in Bonnisseau, Florig and Jofré [3]. Let $G^1, \ldots, G^\mu$ be the connected components of G.

First note that $x_i^h(p, \omega) = 0 \in \mathbb{Q}$, if $(i, h) \notin G$. Fix some $n \in \{1, \ldots, \mu\}$. If $G^n = \{i, h\}$, this means that the i-th consumer consumes only the h-th commodity and he is the only one who consumes this commodity. In that case, $x_i^h(p, \omega) = \sum_{i\in I} \omega_i^h \in \mathbb{Q}$. If G^n has more than two elements, then we associate the rational number $w_i(p, \omega) = p \cdot \omega_i$ to the vertex i and $w_h(p, \omega) = \sum_{i\in I} \omega_i^h \in \mathbb{Q}$ to the vertex h. Since G^n is a finite tree, it has a terminal node. If this node is an element $i \in I$, there exists $h \in L$ such that the edge (i, h) links i to the rest of the tree. This means that the i-th consumer consumes only the h-th commodity. In that case, $x_i^h(p, \omega) = \frac{w_i(p,\omega)}{p_h} \in \mathbb{Q}$. Then, we consider the sub-tree obtained from G^n be deleting the vertex i and the edge (i, h) and we replace $w_h(p, \omega)$ by $w_h(p, \omega) - \frac{w_i(p,\omega)}{p^h} \in \mathbb{Q}$. If the terminal node is an element $h \in L$, there exists $i \in I$ such that the edge (i, h) links h to the rest of the tree. This means that the h-th commodity is only consumed by the i-th consumer. In that case, $x_i^h(p,\omega) = w_h(p,\omega) \in \mathbb{Q}$ and we consider the sub-tree obtained from G^n be deleting the vertex h and the edge (i,h) and we replace $w_i(p,\omega)$ by $w_i(p,\omega) - p^h w_h(p,\omega) \in \mathbb{Q}$. In the two cases, one obtains a sub-tree with one vertex less. Consequently, in a finite number of steps, one checks that for all $(i,h) \in I \times L$, $x_i^h \in \mathbb{Q}$. ∎

Proof of Theorem 2.3. Let $(L_r)_{r=1}^k$ be the partition of commodities associated with the hierarchic equilibrium, i.e. for all $r \in \{1,...,k\}$, $L_r = \operatorname{supp}(p_r)$.

In order to be able to apply results on linear exchange economies established in Gale [14], Bonnisseau, Florig and Jofré [3],[4] and Florig [10], we need to transform the economy into an auxiliary one satisfying conditions (I) - (III).

Step 1. Construction of an auxiliary economy

For every $i \in I$, let $(\omega_{i_1},...,\omega_{i_k}) \in (\mathbb{R}^\ell_+)^k$ such that $\omega_i = \sum_{r=1}^k \omega_{i_r}$ with $\operatorname{supp}(\omega_{i_r}) \subset L_r$ for every $r \in \{1,...,k\}$. For every $r \in \{1,...,k\}$, let

$$I_r^- = \{i \in I \mid p_r \cdot x_i < p_r \cdot \omega_i\};$$

$$I_r^+ = \{i \in I \mid p_r \cdot x_i > p_r \cdot \omega_i\};$$

$$I_r^0 = \{i \in I \mid p_r \cdot x_i = p_r \cdot \omega_i\}$$

and for every $i \in I_r^-$, let

$$g_{i_r} = \omega_{i_r}(p_r \cdot x_i / p_r \cdot \omega_i),$$

for every $i \in I_r^+$,

$$g_{i_r} = \omega_{i_r} + \frac{p_r \cdot x_i - p_r \cdot \omega_i}{\sum_{i \in I_r^+}(p_r \cdot x_i - p_r \cdot \omega_i)} \sum_{i \in I_r^-} (\omega_{i_r} - g_{i_r}).$$

For $i \in I_r^0$, let $g_{i_r} = \omega_{i_r}$. For every $i \in I$, let $g_i = \sum_{r=1}^k g_{i_r}$. Then, $((x_i, w_i)_{i \in I}, \mathcal{P})$ is a hierarchic equilibrium of the economy $\mathcal{L}(\mathbb{R}^\ell_+, b_i, g_i)_{i \in I}$. For every $i \in I$, let $(x_{i_1},...,x_{i_k}) \in (\mathbb{R}^\ell_+)^k$ such that $x_i = \sum_{r=1}^k x_{i_r}$ with $\operatorname{supp}(x_{i_r}) \subset L_r$ for every $r \in \{1,...,k\}$ and let $(b_{i_1},...,b_{i_k}) \in (\mathbb{Q}^\ell_+)^k$ such that $b_i = \sum_{r=1}^k b_{i_r}$ with $\operatorname{supp}(b_{i_r}) \subset L_r$ for every $r \in \{1,...,k\}$. We will further modify the economy $\mathcal{L}(\mathbb{R}^\ell_+, b_i, e_i)_{i \in I}$. For every $r \in \{1,...,k\}$, we make a copy $I_r = \{1_r,...,m_r\}$ of I and with every $i_r \in I_r$, we associate $(x_{i_r}, b_{i_r}, g_{i_r})$. For any $(\mu_r)_{r=1}^k >> 0$, $((x_{i_r}), \sum_{r=1}^k \mu_r p_r)$ is a Walras equilibrium of the economy

$\mathcal{L}(I\!R^{\ell}_{+}, b_{i_r}, g_{i_r})_{i_r \in I_r, r=1,\ldots,k}$. Indeed, since demand equals supply, it only remains to check that everybody consumes a maximal element within his budget set. Let $q = \sum_{r=1}^{k} \mu_r p_r$. For all $r \in \{1, \ldots, k\}$, for all $i_r \in I_r$, $\mathrm{supp}(b_{i_r}) \subset L_r$ and $\mathrm{supp}(g_{i_r}) \subset L_r$. So $q \cdot x_{i_r} = \mu_r p_r \cdot x_{i_r} = \mu_r p_r \cdot g_{i_r} = q \cdot g_{i_r}$. If $b_{i_r} = 0$, trivially x_{i_r} is in the demand of consumer i_r. Otherwise, since consumer i maximize utility at a hierarchic equilibrium, if for some $h' \in L_r$, $b_{i_r}^{h'}/p_r^{h'} < \max_{h \in L_r}(b_{i_r}^{h'}/p_r^{h})$, then we must have $x_{i_r}^{h'} = 0$. ■

Step 2. Ensuring Conditions (I) - (III)

For every $r \in \{1, ..., k\}$ and every $i \in I_r$ such that $b_{i_r} = 0$ we make $\#L_r$ copies of the consumer. We denote these copies $\{i_{rh} \mid h \in L_r\}$ and

$$B_r = \{i_{rh} \mid i_r \in I_r, b_{i_r} = 0, h \in L_r\}$$

the set of the copies of all agents for a given $r \in \{1, ..., k\}$.

Remark that for all r, $i \in I_r^-$, implies $b_{i_r} = 0$, since otherwise, consumer i would not maximize utility at the hierarchic equilibrium. Hence, B_r contains the consumers $i_r \in I_r^-$. For every $i_{rh} \in B_r$, we let i_{rh} be an auxiliary consumer only interested in the commodity $h \in L_r$. For $i_{rh} \in B_r$, let $a_{i_{rh}} \in I\!R^{\ell}$ be a vector having a one at the entry $h \in L_r$ and zeros elsewhere and let

$$e_{i_{rh}} = \begin{cases} g_{i_r} \frac{p_r^h x_{i_r}^h}{p_r \cdot x_{i_r}} & \text{if } p_r \cdot x_{i_r} \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

For $i_r \in I_r$ such that $b_{i_r} \neq 0$ set $a_{i_r} = b_{i_r}$ and $e_{i_r} = g_{i_r}$. Let

$$A_r = \{i \in B_r \mid e_i \neq 0\} \cup \{i_r \in I_r \mid b_{i_r} \neq 0, e_{i_r} \neq 0\}.$$

Now, let $A = \cup_{r=1}^{k} A_r$ and consider the economy $\mathcal{L}(I\!R^{\ell}_{+}, a_i, e_i)_{i \in A}$.

Of course we need a way to come back to the original economy, therefore for every $i \in I$, let $A(i)$ be the set of elements $j \in A$, such that j was derived from consumer i.

Let $(\xi_j)_{j \in A}$ be defined as follows. Let $i \in I$ such that $j \in A(i)$. For some $r \in \{1, ..., k\}$, we have $j \in A_r$. If$j \in A_r \setminus B_r$, then $\xi_j = x_{i_r}$ with i_r as in Step 1. Otherwise, if $j \in B_r$, then there exists a unique $h' \in L_r$ such that $e_j^{h'} > 0$. In fact, $x_i^{h'} = e_j^{h'}$ and we set $\xi_j^{h'} = x_i^{h'}$ and for all $h \neq h'$ we set $\xi_i^h = 0$. Remember that a_j is non-zero only in the h-th coordinate. Now, $((\xi_i)_{i \in A}, \sum_{r=1}^{k} \mu_r p_r)$ is a Walras equilibrium of the economy

$\mathcal{L}(I\!R^\ell_+, a_i, e_i)_{i\in A}$ for any $(\mu_r)_{r=1}^k >> 0$. The argument for this is similar to the one in the previous step. In fact, we simply deleted from the previous economy consumers with zero endowment and we replaced consumers i_r with $b_{i_r} = 0$ by auxiliary consumers who obviously satisfy utility maximization. It is now easy to check that the economy $\mathcal{L}(I\!R^\ell_+, a_i, e_i)_{i\in A}$ satisfies conditions (I) - (II). It also satisfies condition (III) (which by Ref. 14 is equivalent to the existence of a Walras equilibrium under (I) and (II)). ■

Step 3. Rational equilibria

We will now construct a sequence of rational equilibria.

Let $\mathcal{W}_c \subset (I\!R^\ell_+)^{\#A}$ be the set of initial endowments $(\omega'_i)_{i\in\mathcal{I}} \in (I\!R^\ell_+)^{\#A}$ such that:

(i) $\mathrm{supp}(\omega'_i) = \mathrm{supp}(e_i)$ for all $i \in A$;

(ii) $\sum_{i\in A} \omega'_i = \sum_{i\in I} \omega_i$.

By Propositions 3.1 and 5.1 of Bonnisseau, Florig and Jofré [3], Corollary 4.2 of Florig [10] and Michael's selection theorem, there exist continuous selections

$$\pi : \mathcal{W}_c \to I\!R^\ell_{++} \quad \text{and} \quad \xi : \mathcal{W}_c \to (I\!R^\ell_+)^{\#A}$$

of the equilibrium price and allocation correspondences. Furthermore, they may be chosen such that $\pi((e_i)_{i\in A}) = \sum_{r=1}^k p_r$ and $\xi((e_i)_{i\in A}) = (\xi_i)_{i\in A}$. Indeed, a lower semi-continuous correspondence stays lower semi-continuous, if one replaces the image in some point by a point of the image.

Remember that for all $r \in \{1, \ldots, k\}$, $I_r^+ \neq \emptyset$ implies $I_r^- \neq \emptyset$ and for all $i \in I_r^-$, $b_i = 0$. So if $I_r^+ \neq \emptyset$, then $B_r \neq \emptyset$. Since $(\omega_i)_{i\in I} \in (Q^\ell_+)^m$ it is possible to choose for every $n \in I\!N$, some $(e_i^n)_{i\in\mathcal{I}} \in (Q^\ell_+)^{\#A}$ such that for all $i \in A_r \setminus B_r$ with $i \in A(j)$,

$$e_j \geq e_i^n \geq \omega_j,$$

moreover

$$\sum_{i\in A} e_i^n = \sum_{i\in A} e_i = \sum_{i\in I} \omega_i \text{ and } \sum_{i\in A} \| e_i^n - e_i \| < 1/n,$$

and for all $i \in A$

$$\mathrm{supp}(e_i^n) = \mathrm{supp}(e_i.)$$

Let $(\xi_i^n)_{i\in A} = \xi((e_i^n)_{i\in A})$. Then, obviously, $((\xi_i^n)_{i\in A})_{n\in I\!N}$ converges to $(\xi_i)_{i\in A}$ as n goes to infinity.

By Lemma 4.1, for every n we may choose $\{x_1^n, \ldots, x_\nu^n\} \subset (\mathbb{Q}^\ell)^m \cap X(a_i, e_i^n)_{i\in A}$, $(\lambda_s^n)_{s=1}^\nu \in \mathbb{Q}_+^\nu$ with $\sum_{s=1}^\nu \lambda_s^n = 1$, letting $y^n = \sum_{s=1}^\nu \lambda_s^n x_s^n$ such that

$$\| y^n - \xi^n \| \leq 1/n.$$

For every n, set $\sum_{r=1}^k p_r^n = \pi(a_i, e_i^n)_{i\in A}$ with $\mathrm{supp}(p_r^n) \subset L_r$. Note that for every n, $((y_i^n)_{i\in A}, \sum_{r=1}^k \mu_r p_r^n)$ is a Walras equilibrium of $\mathcal{L}(\mathbb{R}_+^\ell, a_i, e_i^n)_{i\in A}$ for any $(\mu_r)_{r=1}^k >> 0$. ■

Step 4. Back to the original economy

For every $n \in \mathbb{N}$, let

$$x_i^n = \sum_{j\in A(i)} y_j^n, \quad \omega_i^n = \sum_{j\in A(i)} e_j^n.$$

Hence, for for every $n \in \mathbb{N}$, $(x_i^n)_{i\in I} \in (\mathbb{Q}_+^\ell)^m$.

Then, for every $n \in \mathbb{N}$ there exists some $\phi(n) \in \mathbb{N}$, such that for every $i \in I$, $x_i^n \in X^{\phi(n)}$. Since $z \in X^n$ implies that $z \in X^{nn'}$ for all $n' \in \mathbb{N}$, we can choose $\phi(n)$, such that $\phi : \mathbb{N} \to \mathbb{N}$ is strictly increasing.

Let $\varphi : \mathbb{N} \to \mathbb{N}$ be strictly increasing such that for every $n \in \mathbb{N}$,

$$\frac{1}{\phi(n)} \min_{r,r'\in\{1,\ldots,k\}} \min_{(h,h')\in \mathrm{supp}(p_r^n)\times \mathrm{supp}(p_{r'}^n)} \left(\frac{p_r^{hn}}{p_{r'}^{h'n}}\right) > \frac{2}{\varphi(n)} \| \sum_{i\in I} \omega_i \| .$$

For every $n \in \mathbb{N}$, let $\mathcal{P}^n = (p_1^n, \ldots, p_k^n)$ where for every $r \in \{1, \ldots, k\}$, p_r^n is as defined above. Let

$$p^n = \sum_{r=1}^k \frac{(1/\varphi(n))^{r-1}}{\| \sum_{r=1}^k (1/\varphi(n))^{r-1} p_r^n \|} p_r^n \text{ and } w_i^n = \max\{p^n \cdot x_i^n, p^n \cdot \omega_i\}. ■$$

Step 4. *For any $n \in \mathbb{N}$, $((x_i^n, w_i^n)_{i\in I}, p^n)$ is a dividend equilibrium of the economy $\mathcal{L}(X_i^{\phi(n)}, b_i, \omega_i)_{i\in I}$.*

Obviously, $\sum_{i\in I} x_i^n = \sum_{i\in I} \omega_i$ and for all $i \in I$, $x_i^n \in B_i(X^{\phi(n)}, p^n, w_i^n)$. For all $i \in I$, let

$$r_i = \min\{r \in \{1, \ldots, k\} \mid \mathrm{supp}(\omega_i + x_i^n) \cap L_r \neq \emptyset\}.$$

Note that the choice of φ implies that for all $z \in B_i(X^{\phi(n)}, p^n, w_i^n)$ and for all $h \in \cup_{r=1}^{r_i-1} L_r$, $z_h = 0$. Thus, goods in $\cup_{r=1}^{r_i-1} L_r$ are inaccessible to consumer i.

Since

$$r_i = \min\{r \in \{1,\ldots,k\} \mid \mathrm{supp}(\omega_i + x_i) \cap L_r \neq \emptyset\}$$

and $x_i \in D_i(I\!R^\ell_+, \mathcal{P}, w_i)$ we have for all $h \in \cup_{r=r_i+1}^k L_r$, $b_i^h = 0$.

By the choice of φ, for no $i \in I$, it would be worthwile changing the allocation x_i^n on commodities in $h \in \cup_{r=r_i+1}^k L_r$. On the one hand, buying more of these commodities is pointless since they yield utility zero to him. On the other hand, consuming less does not yield enough income to enable the consumer to buy more commodities in L_{r_i}. In order, to conclude that for all $n \in I\!N$ and all $i \in I$, $x_i^n \in D_i(X^{\phi(n)}, p^n, w_i^n)$, it is sufficient to show that all $i \in I$ maximize utility on the economy restricted to L_{r_i} (and income equal to w_i^n). If $b_i^h = 0$ for all $h \in L_{r_i}$ this is trivial. If for some $h \in L_{r_i}$, $b_i^h > 0$, then to this consumer there corresponds a consumer $j \in A$ such that $e_j^{hn} \geq e_j^h \geq \omega_i^h$ for every $h \in L_{r_i}$ and j consumes a maximal element y_j in his budget set. Since $x_i^n = \sum_{j' \in A(i)} y_{j'}^n$ and x_i^n coincides with y_j^n on L_{r_i}, we may conclude that $x_i^n \in D_i(X^{\phi(n)}, p^n, w_i^n)$. Therefore, $((x_i^n, w_i^n)_{i \in I}, p^n)$ is a dividend equilibrium. ∎

Part (iii) of the theorem may be proven by exactly the same arguments as in Florig [9] working with the sequence (x^n, p^n) instead of perturbed equilibria. ∎

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