Walrasian equilibrium as a limit of competitive equilibria without divisible goods*

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Abstract

This paper investigates the limit properties of a sequence of competitive outcomes existing for economies where all commodities are indivisible, as indivisibility vanishes. The nature of this limit depends on whether the "strong survival assumption" is assumed or not in the limit economy, a standard "convex economy". If this condition holds, then the equilibrium sequence converges to a Walras equilibrium for the convex economy; otherwise it converges to a hierarchic equilibrium, a competitive outcome existing in this economy despite the fact that a Walras equilibrium might not exist.

Keywords: competitive equilibrium, indivisible goods, convergence.

JEL Classification: C62, D50, E40

1 Introduction

The "discrete economy" proposed by Florig and Rivera [13] is a private ownership economy where the indivisibility of commodities matters at an individual level, but is negligible at the level of the entire economy. The continuum of individuals that participate in this economy is partitioned into a finite number of types of agents. Individually, consumption and production sets are discrete sets (the same subset for agents of the same type), while their aggregate by type of agent is the convex hull of the individual set. Consumers of a given type are identical, except for a continuum parameter with which we initially endow them. This parameter could be identified as "fiat money" (see Drèze and Müller [8]), whose sole role is to facilitate the trade of indivisible commodities.

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Despite the fact that fiat money has no intrinsic value whatsoever, since it does not enter into consumer preferences, it plays a fundamental role in the assignment of resources.[1]

Under mild conditions, Florig and Rivera [13] prove the existence of a "rationing equilibrium" for discrete economies, a competitive outcome in which fiat money has a strictly positive price.[2] Moreover, when the distribution of fiat money is such that different consumers are initially endowed with a different amount of it, then a rationing equilibrium is a "Walras equilibrium with fiat money" for the discrete economy.[3] The proof of these results uses a "weak survival condition", i.e., the initial endowment of resources belongs to the convex hull of the consumption set.

The aim at this paper is to investigate the limit properties of rationing equilibrium as indivisibility vanishes. This limit is an element of a "convex economy" (with fiat money), a standard economy with a finite number of agents, where both consumption and production sets are polyhedral, and where consumers are initially endowed with fiat money. Since a convex economy as can always be approximated by a sequence of discrete economies,[4] the question in this paper actually refers to the limit properties of a rationing equilibrium sequence, whose elements belong to a sequence of discrete economies that converges to a convex economy.

Our main finding is that the nature of this limit is strongly dependent on the type of survival condition assumed in the convex economy. Under a "strong survival condition" the limit is a Walras equilibrium with fiat money for the convex economy. When fiat money has a strictly positive price in the convex economy, then the Walras equilibrium with fiat money corresponds to a "dividend equilibrium" (or "equilibrium with slack"), a generalized notion of the Walras equilibrium that allows for the possibility that some agents spend more than the value of their initial endowment (see Kajii [21] and Mas-Colell [23]). A situation like this may occur when, for instance, local satiation holds for some consumers, or when some price rigidities are present in the convex economy. Otherwise, fiat money becomes worthless in the convex economy (its price is zero at equilibrium), thus the Walras equilibrium with fiat money is a standard Walras equilibrium.

In our opinion, a most interesting situation occurs when a "weak survival assumption" is assumed for the convex economy. When the initial endowment of resources of each consumer does not belong to the interior of consumption set, the indivisibility of commodities could matter, re-

[1]Fiat money should not be confused with "commodity money" (also known as "inside money"), yet another continuum parameter widely employed in the literature to assure the existence of a Walras equilibrium when commodities are indivisible (see Bobzin [5] for a review of general equilibrium models with indivisible commodities). Contrary to fiat money, commodity money satisfies overriding desirability, i.e. it is so desirable by the agents that an adequate amount of it could replace the consumption of any bundle of indivisible commodities.

[2]The efficiency and core equivalence properties of a rationing equilibrium are studied in Florig and Rivera [12].

[3]In the case of a finite number of consumers and indivisibility of goods, Henry [19] shows that a Walras equilibrium may not exist, while Shapley and Scarf [30] show that even the core may be empty. In the case of a continuum of agents and perfect divisibility of goods ("large economy"), Aumann [3], Dierker [7] and Hildenbrand [20], among others, study the existence of equilibrium and related properties. Contrary to large economies, the indivisibility of commodities at an individual level is a crucial condition that remains active for discrete economies.

[4]In doing this, the types of agents of these discrete economies are the agents of the convex economy, and the corresponding individual consumption and production sets are discrete subsets, such that their convex hulls converge to the polyhedral sets of the standard economy. The Kuratowski – Painlevé set convergence notion is used in this paper (see Rockafellar and Wets [28]).

gardless of how small it is. It may then occur that not all consumers have access to all goods, i.e., a commodity may be so expensive that some consumers who do not own expensive goods would not be able to purchase a single unit by selling their entire initial endowment. When the consumption goods become "more divisible", i.e., if the minimal unit per commodity decreases, then the equilibrium price may react so that the situation persists.

Following Gay [14], based on a generalized concept of price, several authors have proposed generalizations of the Walras equilibrium existing in the convex case, even when the Walras equilibrium does not exist due to a failure of the strong survival assumption (see, for instance, Danilov and Sotskow [6], Marakulin [25] and Mertens [26]). Backed up by several examples, Florig [10] proposes an interpretation of those generalized prices in terms of small indivisibilities, introducing the concept of "hierarchic equilibrium".[5]

The main result of this paper is the proof of that when the weak survival condition holds in the convex economy, then the rationing equilibrium sequence converges to a hierarchic equilibrium of that economy. This result formalizes the interpretation of hierarchic equilibria in terms of small indivisibilities given in Florig [10].

A direct consequence of our convergence results is that when both the strong survival assumption and the local non-satiation hypothesis are not satisfied, then a Walras equilibrium does not exist for a convex economy.[6] We highlight, however, that this fact certainly does not inhibit the possibility of the existence of a Walras equilibrium, or a related competitive outcome, in convex economies maintaining one of these conditions, while relaxing the other. For instance, using variants of the "irreducibility" condition introduced by Gale [15, 16], which is a weaker condition than the strong survival condition, but stronger than the weak survival condition, several authors have studied the existence of competitive outcomes in economy (see Baldrya and Ghosalb [4], Florig [9], Gottardi and Hens [17], Hammond [18] and McKenzie [22], Spivak [31], among others). The existence of competitive outcomes for a convex economy when the non-satiation condition is relaxed was studied in, for instance, Drèze and Müller [8], Marakov [24] and Sato [29]. It is worth mentioning that all of these contributions maintain the perfect divisibility of goods.

[5]Broadly speaking, a hierarchic equilibrium might be interpreted as a sort of "vectorial" Walras equilibrium, where each "component" is a Walras equilibrium for a certain group of agents. If there is only one group, then the hierarchic equilibrium is a Walras equilibrium. When the weak survival condition holds in the convex economy, these groups arise endogenously due to the fact that some agents cannot survive in the economy, since they are unable to buy goods that are beyond their possibilities. However, nothing prevents that individuals in similar conditions from actually trading certain goods that are affordable to them. Identifying these groups as "socio-economic classes" in terms of individual wealth, so that the wealth of different classes is of a different order of magnitude, at a rationing equilibrium poorer consumers do not have access to all the expensive commodities that the richer have access to. Piccioni and Rubinstein [27] also provide an alternative interpretation of the hierarchic equilibrium in terms of parallel currencies existing in the economy, ordered by a strict hierarchy and where the trade between currencies of different ranks is prohibited. A low-ranked currency can only be used in a smaller set of markets than a highly-ranked currency. Finally, in the case of linear preferences, Florig [11] shows that a hierarchic equilibrium is the limit of standard competitive equilibria of economies with discrete consumption sets converging to the positive orthant.

[6]This result follows directly from the fact that every Walras equilibrium in the convex economy can be approximated by a rationing equilibrium sequence from a discrete sequence that converges to the convex economy. See §4 further on in this paper.

This paper is organized as follows. In §2 we introduce preliminary concepts and notions, while §3 presents the model of economies and equilibria notions used in this paper. Therein we also define the notion of convergence of a sequence of discrete economies to a convex economy. In §4 we present the main contributions of this paper, the convergence of equilibrium results (namely, Proposition 4.2 when the strong survival condition holds, and Theorem 4.1 for a general case). Finally, most of the proofs are provided in the Appendix, i.e., §5.

2 Notation and some concepts

In what follows, 0_m is the origin of $\mathbb{R}^m$, x^{t} is the transpose of $x \in \mathbb{R}^m$, whose Euclidean norm is $\|x\|$; the inner product between $x, y \in \mathbb{R}^m$ is $x \cdot y = x^{\mathrm{t}} y$, and the open ball centered at x with radius $\varepsilon > 0$ is $\mathbb{B}(x, \varepsilon)$. For a couple of sets K_1, $K_2 \subseteq \mathbb{R}^m$, $\xi \in \mathbb{R}$ and $p \in \mathbb{R}^m$, we denote $\xi K_1 = \{\xi x : x \in K_1\}$, $p \cdot K_1 = \{p \cdot x : x \in K_1\}$ and $K_1 \pm K_2 = \{x_1 \pm x_2 : x_1 \in K_1, x_2 \in K_2\}$, while the set-difference between them is denoted $K_1 \setminus K_2$. Furthermore, $\operatorname{cl} K_1$, $\operatorname{int} K_1$ and $\operatorname{conv} K_1$ denote, respectively, the closure, interior and the convex hull of K_1.

In the following, $\lambda(\cdot)$ denotes the standard Lebesgue measure in the underlying space, and for a couple of sets $K_1 \subseteq \mathbb{R}^m$ and $K_2 \subset \mathbb{R}^n$, $L^1(K_1, K_2)$ stands for the subset of Lebesgue integrable functions from K_1 to K_2.

We follow Rockafellar and Wets [28] to denote

$$\mathbb{N}_\infty = \{\mathbf{N} \subseteq \mathbb{N} : \mathbb{N} \setminus \mathbf{N} \text{ is finite}\} \qquad \text{and} \qquad \mathbb{N}_\infty^* = \{\mathbf{N} \subset \mathbb{N} : \mathbf{N} \text{ is infinite}\},$$

and for $\mathbf{N} \in \mathbb{N}_\infty^*$, the subset of accumulation points of $\{x_n\}_{n \in \mathbf{N}}$ is[7]

$$\operatorname{acc}\{x_n\}_{n \in \mathbf{N}} \quad = \quad \{x \in \mathbb{R}^m : \exists \mathbf{N}' \subset \mathbf{N},\, \mathbf{N}' \in \mathbb{N}_\infty^*,\, x_n \to_{\mathbf{N}'} x\}.$$

We also recall that the outer limit of a sequence of subsets $\{K_n\}_{n \in \mathbb{N}}$ of $\mathbb{R}^m$ is the subset

$$\limsup_{n \to \infty} K_n \quad = \quad \{x \in \mathbb{R}^m : \exists \mathbf{N} \in \mathbb{N}_\infty^*,\ \exists x_n \in K_n,\, n \in \mathbf{N}, \text{ with } x_n \to_{\mathbf{N}} x\},$$

while the inner limit of the sequence is

$$\liminf_{n \to \infty} K_n \quad = \quad \{x \in \mathbb{R}^m : \exists \mathbf{N} \in \mathbb{N}_\infty,\ \exists x_n \in K_n, n \in \mathbf{N},\ \text{with } x_n \to_{\mathbf{N}} x\}.$$

The sequence of subsets $\{K_n\}_{n \in \mathbb{N}}$ of $\mathbb{R}^m$ converges in the sense of Kuratowski – Painlevé to the subset $K \subseteq \mathbb{R}^m$ if

$$\limsup_{n \to \infty} K_n = \liminf_{n \to \infty} K_n = K,$$

[7]For $\mathbf{N} \in \mathbb{N}_\infty$ or $\mathbf{N} \in \mathbb{N}_\infty^*$, and a sequence $\{x_n\}_{n \in \mathbb{N}}$ of vectors of $\mathbb{R}^m$, we write $x_n \to_{\mathbf{N}} x$ when $\lim_{n \to \infty, n \in \mathbf{N}} x_n = x$; in case $\mathbf{N} = \mathbb{N}$ we put $x_n \to x$.

in which case we write $\lim_{n\to\infty} K_n = K$.

Finally, the outer limit of a correspondence $\Psi : \mathbb{R}^m \rightrightarrows \mathbb{R}^k$ at $\bar{x} \in \mathbb{R}^m$ is

$$\limsup_{x\to\bar{x}} \Psi(x) = \bigcup_{\{x_n\to\bar{x}\}} \limsup_{n\to\infty} \Psi(x_n).$$

3 The model

3.1 Economies and convergence

A "convex economy" (with fiat money) is a private ownership economy, which has a finite number L of commodities, and a finite number of consumers and producers (indexed by I and J, respectively). Each firm $j \in J$ is characterized by a production set $Y_j \subset \mathbb{R}^L$, and each consumer $i \in I$ is characterized by a consumption set $X_i \subset \mathbb{R}^L$, a vector of initial resources $e_i \in \mathbb{R}^L$, a strict preference correspondence $P_i : X_i \rightrightarrows X_i$, and an initial amount $m_i \in \mathbb{R}_{++}$ of fiat money. This parameter has no intrinsic value whatsoever. The total initial resources of the economy is $e = \sum_{i\in I} e_i \in \mathbb{R}^L$, and for $(i,j) \in I \times J$, $\theta_{ij} \in [0,1]$ is the consumer i's share in firms j. As usual, we assume that for every $j \in J$, $\sum_{i\in I} \theta_{ij} = 1$.

The convex economy (with fiat money) is then the collection

$$\mathcal{E} \quad = \quad \left(\{X_i, P_i, e_i\}_{i\in I}, \{Y_j\}_{j\in J}, \{\theta_{ij}\}_{(i,j)\in I\times J}, \{m_i\}_{i\in I}\right).$$

We now proceed to define a sequence of "discrete economies" that approximates the convex economy as indivisibility vanishes. For the sequel we use given sequences $\nu_h : \mathbb{N} \to \mathbb{N}$, $h = 1, \dots, L$, such that $\lim_{n\to\infty} \nu_h(n) = \infty$, for all h. The family of subsets $\{M^n\}_{n\in\mathbb{N}}$ with

$$M^n \quad = \quad \left\{\xi = (\xi_1, \dots, \xi_L)^{\mathrm{t}} \in \mathbb{R}^L : (\nu_1(n)\xi_1, \dots, \nu_L(n)\xi_L)^{\mathrm{t}} \in \mathbb{Z}^L\right\},$$

then converges in the sense of Kuratowski-Painlevé to $\mathbb{R}^L$. Let $\{X_i^n\}_{n\in\mathbb{N}}, i \in I$, and $\{Y_j^n\}_{n\in\mathbb{N}}, j \in J$, such that

$$Y_j^n = Y_j \cap M^n \neq \emptyset \quad \text{and} \quad X_i^n = X_i \cap M^n \neq \emptyset.$$

Each discrete economy of the sequence is conformed by a continuum of consumers and producers, they partitioned into finitely many types of agents, namely consumers and producers of the convex economy. We assume that agents of type $i \in I$ and $j \in J$ are indexed, respectively, by compact subsets $T_i \subset \mathbb{R}$ and $T_j \subset \mathbb{R}$, pairwise disjoint. The subset of consumers and firms of a discrete economy is respectively denoted by

$$\mathcal{I} = \bigcup_{i\in I} T_i \qquad \text{and} \qquad \mathcal{J} = \bigcup_{j\in J} T_j.$$

The type of producer $t \in \mathcal{J}$ is $j(t) \in J$, and firms of type $j \in J$ of the discrete economy indexed by $n \in \mathbb{N}$ are characterized by the production set $Y_j^n \subset \mathbb{R}^L$. The aggregate production set of these firms is the convex hull of $\lambda(T_j)Y_j^n$, a production plan for firm $t \in \mathcal{J}$ is denoted by $y(t) \in Y_{j(t)}^n$, and the set of admissible production plans is

$$Y^n = \left\{ y \in L^1(\mathcal{J}, \cup_{j \in J} Y_j^n) : y(t) \in Y_{j(t)}^n \text{ a.e. } t \in \mathcal{J} \right\}.$$

The type of consumer $t \in \mathcal{I}$ is $i(t) \in I$, and each consumer of type $i \in I$ of the discrete economy indexed by $n \in \mathbb{N}$ is characterized by the consumption set $X_i^n \subset \mathbb{R}^L$, an initial endowment of resources $e_i \in \mathbb{R}^L$ and a strict preference correspondence $P_i^n : X_i^n \rightrightarrows X_i$, the restriction of P_i to X_i^n. A consumption plan of individual $t \in \mathcal{I}$ is denoted by $x(t) \in X_{i(t)}^n$, and the set of admissible consumption plans is

$$X^n = \left\{ x \in L^1(\mathcal{I}, \cup_{i \in I} X_i^n) : x(t) \in X_{i(t)}^n \text{ a.e. } t \in \mathcal{I} \right\}.$$

The total initial resources of the economy is $e = \sum_{i \in I} \lambda(T_i)\, e_i \in \mathbb{R}^L$, and for $(i,j) \in I \times J$, $\theta_{ij} \geq 0$ is the consumer of type i's share in firms of type j. For every $j \in J$, we assume that $\sum_{i \in I} \lambda(T_i)\theta_{ij} = 1$. In addition, we also assume that each consumer $t \in \mathcal{I}$ is initially endowed with an amount of fiat money $m(t) \in \mathbb{R}_+$, where $m \in L^1(\mathcal{I}, \mathbb{R}_+)$.[8]

A discrete economy indexed by $n \in \mathbb{N}$ is the collection

$$\mathcal{E}^n \quad = \quad \left(\{X_i^n, P_i^n, e_i\}_{i \in I}, \{Y_j^n\}_{j \in J}, \{\theta_{ij}\}_{(i,j) \in I \times J}, m, \{T_i\}_{i \in I}, \{T_j\}_{j \in J} \right),$$

and we say that the sequence of economies $\{\mathcal{E}^n\}_{n \in \mathbb{N}}$ approximates economy $\mathcal{E}$.

The feasible consumption-production plans of $\mathcal{E}^n$ are the elements of[9]

$$A(\mathcal{E}^n) \quad = \quad \left\{ (x,y) \in X^n \times Y^n : \int_{\mathcal{I}} x(t)dt = \int_{\mathcal{J}} y(t)dt + e \right\}.$$

3.2 Demand and supply

We now define supply and demand concepts for economy $\mathcal{E}^n$; the extension to economy $\mathcal{E}$ is direct. In what follows $p \in \mathbb{R}^L$, $q \in \mathbb{R}_+$ and K stands for a "salient cone" of $\mathbb{R}^L$, whose family is $\mathcal{C}_L$.[10] The "profit", the "Walras supply" and the "rationing supply" of a type $j \in J$ firm are, respectively,

$$\pi_j^n(p) = \lambda(T_j) \sup_{z \in Y_j^n} p \cdot z, \quad S_j^n(p) = \operatorname*{arg\,max}_{z \in Y_j^n} p \cdot z$$

[8]The conditions for all $i \in I$, $m(t) = m_i \in \mathbb{R}_+$, a.e. $t \in T_i$, and $\lambda(T_i) = \lambda(T_j) = 1$, $(i,j) \in I \times J$, might be assumed just for interpretative purposes. These conditions are not required to prove our main results.

[9]See Aubin and Frankowska [2] for the definition of integral of a correspondence.

[10]We recall a convex set $K \subset \mathbb{R}^L$ is a convex cone if $0_L \in K$ and $\xi K \subset K$ for all $\xi > 0$; a convex cone K is said to be "salient" if $K \cap -K = \{0_L\}$.

and

$$\sigma_j^n(p, K) = \left\{z \in S_j^n(p) \,:\, p \neq 0_L \Rightarrow (Y_j - \{z\}) \cap K = \{0_L\}\right\}.$$

The "income" of consumer $t \in \mathcal{I}$ is

$$w_t^n(p, q) = p \cdot e_{i(t)} + qm(t) + \sum_{j \in J} \theta_{i(t)j} \pi_j^n(p),$$

whose "budget set" is $B_t^n(p, q) = \left\{\xi \in X_{i(t)} : p \cdot \xi \leq w_t^n(p, q)\right\}$.

The "Walras demand", the "weak demand" and "rationing demand" for consumer $t \in \mathcal{I}$ are, respectively,

$$d_t^n(p, q) = \left\{\xi \in B_t^n(p, q) : B_t^n(p, q) \cap P_{i(t)}^n(\xi) = \emptyset\right\}, \quad D_t^n(p, q) = \limsup_{(p', q') \to (p, q)} d_t^n(p', q'),$$

and

$$\delta_t^n(p, q, K) = \left\{\xi \in D_t^n(p, q) : P_{i(t)}^n(\xi) - \{\xi\} \subseteq K\right\}.$$

Remark 3.1. *As we shall ensure that $d_t^n(\cdot)$ is closed valued and locally bounded, Theorem 5.19 in Rockafellar and Wets [28] implies that $D_t^n(\cdot)$ is upper hemi-continuous while $d_t^n(\cdot)$ may fail to be upper hemi-continuous. Notice also that, by definition,*

$$d_t^n(p, q) \subseteq D_t^n(p, q) \quad \text{and} \quad \delta_t^n(p, q, K) \subseteq D_t^n(p, q).$$

The next characterization of weak demand is a direct consequence of Proposition 3.1 in Florig and Rivera [13].

Proposition 3.1. *If $m(t) > 0$ and $(p, q) \in \mathbb{R}^L \times \mathbb{R}_{++}$, then the following holds for all $n \in \mathbb{N}$:*

$$D_t^n(p, q) \quad = \quad \left\{\xi \in B_t^n(p, q) : \inf\left\{p \cdot P_{i(t)}^n(\xi)\right\} \geq w_t^n(p, q),\ \xi \notin \operatorname{conv} P_{i(t)}^n(\xi)\right\}.$$

3.3 Equilibrium

Next definition gives some of the equilibrium notions that are used in this paper. They are presented for economy $\mathcal{E}^n$, and their extension to economy $\mathcal{E}$ is direct.

Definition 3.1. *Given $(x_n, y_n, p_n, q_n) \in A(\mathcal{E}^n) \times \mathbb{R}^L \times \mathbb{R}_+$ and $K_n \in \mathcal{C}_L$, we call*

(a) (x_n, y_n, p_n, q_n) a "Walras equilibrium with fiat money" of $\mathcal{E}^n$, if for a.e. $t \in \mathcal{I}$, $x_n(t) \in d_t^n(p_n, q_n)$ and for a.e. $t \in \mathcal{J}$, $y_n(t) \in S_{j(t)}^n(p_n)$,

(b) (x_n, y_n, p_n, q_n) a "weak equilibrium" of $\mathcal{E}^n$, if for a.e. $t \in \mathcal{I}$, $x_n(t) \in D_t^n(p_n, q_n)$ and for a.e. $t \in \mathcal{J}$, $y_n(t) \in S_{j(t)}^n(p_n)$,

(c) $(x_n, y_n, p_n, q_n, K_n)$ *a "rationing equilibrium" of* $\mathcal{E}^n$*, if for a.e.* $t \in \mathcal{I}$, $x_n(t) \in \delta_t^n(p_n, q_n, K_n)$ *and for a.e.* $t \in \mathcal{J}$, $y_n(t) \in \sigma_t^n(p_n, K_n)$.[11]

Remark 3.2. *Note that every Walras equilibrium is a weak equilibrium and every weak equilibrium is a rationing equilibrium.*

We end this part introducing the "hierarchic equilibrium" concept, yet another a competitive outcome for economies $\mathcal{E}^n$ and $\mathcal{E}$. Hereinafter, for $k \in \mathbb{N}$, $[p_1, \ldots, p_k] \in \mathbb{R}^{L\times k}$ is the matrix whose columns are $p_1, \ldots, p_k \in \mathbb{R}^L$. Given that, a "hierarchic price" for consumption goods is $\mathcal{P} = [p_1, \ldots, p_k]^{\mathrm{t}} \in \mathbb{R}^{k\times L}$, and the "hierarchic value" of $\xi \in \mathbb{R}^L$ is $\mathcal{P}\xi = (p_1 \cdot \xi, \ldots, p_k \cdot \xi)^{\mathrm{t}} \in \mathbb{R}^k$. Moreover, denoting by $\sup_{lex}$ the supremum with respect to $\leq_{lex}$, the lexicographic order[12] on $\mathbb{R}^L$, the "hierarchic supply" and the "hierarchic profit" of a firm of type $j \in J$ of economy $\mathcal{E}^n$ at $\mathcal{P}$ are, respectively,

$$S_j^n(\mathcal{P}) = \{z \in Y_j^n : \forall z' \in Y_j^n,\ \mathcal{P}z' \leq_{lex} \mathcal{P}z\} \qquad \text{and} \qquad \pi_j^n(\mathcal{P}) = \lambda(T_j)\sup\nolimits_{lex}\{\mathcal{P}z : z \in Y_j^n\},$$

and given $\mathcal{Q} \in \mathbb{R}_+^k$, the hierarchic budget set of consumer $t \in \mathcal{I}$ is

$$B_t^n(\mathcal{P}, \mathcal{Q}) \;=\; \mathrm{cl}\left\{\xi \in X_{i(t)}^n : \mathcal{P}\xi \leq_{lex} \mathcal{P}e_{i(t)} + m(t)\,\mathcal{Q} + \sum_{j\in J}\theta_{i(t)j}\pi_j^n(\mathcal{P})\right\}.$$

Based in Florig [10], we introduce the next equilibrium concept.[13]

Definition 3.2. *A collection* $(x_n, y_n, \mathcal{P}_n, \mathcal{Q}_n) \in A(\mathcal{E}^n) \times \mathbb{R}^{k\times L} \times \mathbb{R}_+^k$ *is a "hierarchic equilibrium" of the economy* $\mathcal{E}^n$ *if:*

(a) for a.e. $t \in \mathcal{J}$, $y_n(t) \in S_{j(t)}^n(\mathcal{P}_n)$,

(b) for a.e. $t \in \mathcal{I}$, $x_n(t) \in B_t^n(\mathcal{P}_n, \mathcal{Q}_n)$ *and* $P_{i(t)}^n(x_n(t)) \cap B_t^n(\mathcal{P}_n, \mathcal{Q}_n) = \emptyset$.

The number $k \in \mathbb{N}$ in last expressions will be determined at the equilibrium. When $k = 1$, the hierarchic equilibrium reduces to a Walras equilibrium (with fiat money).

[11] The salient cone in the rationing equilibrium definition is determined endogenously as part of the equilibrium, and summarizes the information that each consumer needs to have in addition to market prices (and their own characteristics) in order to formulate a demand, leading to a stable economic situation, in the sense that no further trading can take place making all participants in a second round of trading strictly better off. Under general conditions over the economy, the existence of such an equilibrium is proved in Florig and Rivera [13].

[12] For $(s,t) \in \mathbb{R}^m \times \mathbb{R}^m$, we recall $s \leq_{lex} t$, if $s_r > t_r$, $r \in \{1, \ldots, m\}$ implies that $\exists \rho \in \{1, \ldots, r-1\}$ such that $s_\rho < t_\rho$. We write $s <_{lex} t$ if $s \leq_{lex} t$, but not $t \leq_{lex} s$. The maximum and the argmax with respect to this order are denoted by $\max_{lex}$ and argmax_{lex}, respectively (similarly for $\min_{lex}$ and argmin_{lex}).

[13] Marakulin [25] introduced a similar notion for exchange economies, using non-standard analysis.

4 Hypotheses and convergence results

4.1 Hypotheses

Assumptions below are used at different parts of this paper, they depending on the convergence result to be established.

Assumption C. *For all* $(i, j) \in I \times J$, X_i *and* Y_j *are convex and compact polyhedral sets.*[14]

Assumption P. *For all* $i \in I$, P_i *is irreflexive and transitive, and has an open graph in* $X_i \times X_i$.

Assumption M. $m : \mathcal{I} \to \mathbb{R}_+$ *is bounded and for a.e.* $t \in \mathcal{I}$, $m(t) > 0$.

Assumption S. *For all* $i \in I$, $e_i \in \left(X_i - \sum_{j \in J} \theta_{ij} \lambda(T_j)\, Y_j\right)$.

Assumption SA. *For all* $i \in I$ $e_i \in \operatorname{int} \left(X_i - \sum_{j \in J} \theta_{ij} \lambda(T_j)\, Y_j\right)$.

Assumption A. *For all* $n \in \mathbb{N}$, $i \in I$ *and all* $j \in J$, $X_i = \operatorname{conv} X_i^n$ *and* $Y_j = \operatorname{conv} Y_j^n$.

Assumption F. *For all* $i \in I$ *and each face* F *of* X_i *such that*[15]

$$\left(\{e_i\} + \sum_{j \in J} \theta_{ij} \lambda(T_j) Y_j\right) \cap X_i \subseteq F,$$

the sequence $\{F \cap X_i^n\}_{n \in \mathbb{N}}$ *converges in the sense of Kuratowski-Painlevé to* F.

Assumption **F** requires that X_i^n restricted to the affine subspace for which the interiority assumption holds converges to X_i restricted to that affine subspace. This is important to ensure that the budget set for a sequence of equilibria of the economies $\mathcal{E}^n$ converges to a budget set of the economy $\mathcal{E}$ for some limit of the price sequence considered.

The following proposition is an immediate consequence of Theorem 4.1 in Florig and Rivera [13]. For the proof of that result, it is enough to check that Assumption **C** implies that both the consumption and production sets of any economy of sequence $\{\mathcal{E}^n\}_{n \in \mathbb{N}}$ that approximates $\mathcal{E}$ are finite (i.e., the number of its elements is finite). The proposition ensures that the sequence of equilibria for which we study convergence do actually exist.

Proposition 4.1. *Suppose* $\mathcal{E}$ *satisfies Assumptions* **C**, **P**, **M** *and* **S**, *and let* $\{\mathcal{E}^n\}_{n \in \mathbb{N}}$ *be a sequence of economies approximating* $\mathcal{E}$. *For each* $n \in \mathbb{N}$, *there exists a rationing equilibrium* $(x_n, y_n, p_n, q_n, K_n)$, *with* $q_n > 0$, *for economy* $\mathcal{E}^n$.

[14] That is, the convex hull of a finite number of vectors.

[15] For a convex compact polyhedron $P \subset \mathbb{R}^m$, a face is a set $F \subseteq P$ such that there exists $\psi \in \mathbb{R}^m$ with $F = \operatorname{argmax} \psi \cdot P$.

4.2 Convergence under the "strong survival assumption"

In the next proposition, the strong survival assumption **SA** plays an important role in establishing the convergence to a Walras equilibrium. Despite that this hypothesis is widely used in the literature, it is unrealistic, because it states that every consumer is initially endowed with a strictly positive quantity of every existing good. Typically, most consumers have a single good to sell (usually, their labor). In fact, it implies that all agents have the same level of income at equilibrium in the sense that they have all access to the same goods.

Proposition 4.2. *Suppose $\mathcal{E}$ satisfies Assumptions* **C**, **P**, **M** *and* **SA**, *and let $\{\mathcal{E}^n\}_{n\in\mathbb{N}}$ be a sequence of economies approximating $\mathcal{E}$ satisfying Assumption* **A**. *For each $n \in \mathbb{N}$, let (x_n, y_n, p_n, q_n) be a weak equilibrium of $\mathcal{E}^n$, with $q_n > 0$ and $\| (p_n, q_n) \| = 1$. Then, there exists $\mathbf{N} \in \mathbb{N}^*_\infty$, such that the following hold:*

- *(a)* $(p_n, q_n) \to_{\mathbf{N}} (p^*, q^*)$,
- *(b) there is $(x^*, y^*) \in A(\mathcal{E})$, such that for a.e. $t \in \mathcal{I}$, $x^*(t) \in \operatorname{acc}\{x_n(t)\}_{n\in\mathbf{N}}$, and for a.e. $t' \in \mathcal{J}$, $y^*(t') \in \operatorname{acc}\{y_n(t')\}_{n\in\mathbf{N}}$, with (x^*, y^*, p^*, q^*) a Walras equilibrium with fiat money for $\mathcal{E}$.*

Moreover, if for a.e. $t \in \mathcal{I}$, $x^*(t) \in \operatorname{cl} P_{i(t)}(x^*(t))$, then (x^*, y^*, p^*) is a Walras equilibrium for $\mathcal{E}$.

Proof. First note that $\{\mathcal{E}^n\}_{n\in\mathbb{N}}$ approximating $\mathcal{E}$ implies that for all $i \in I$, $\lim_{n\to\infty} X_i^n = X_i$. By Assumption **SA**, the smallest face of X_i containing

$$\left(\{e_i\} + \sum_{j\in J} \theta_{ij}\lambda(T_j)Y_j\right) \cap X_i$$

is X_i, which implies that Assumption **F** is satisfied. Therefore, all the assumptions of Theorem 4.1 below are satisfied. Assumption **SA** implies that for a hierarchic equilibrium $(x, y, \mathcal{P}, \mathcal{Q})$ with $\mathcal{P} = [\mathrm{p}_1, \ldots, \mathrm{p}_k]^{\mathrm{t}} \in \mathbb{R}^{k\times L}$ and $\mathcal{Q} = (\mathrm{q}_1, \ldots, \mathrm{q}_k)^{\mathrm{t}} \in \mathbb{R}^k_+$ (see definition in next section), such that $(x^*, y^*, \mathrm{p}_1, \mathrm{q}_1)$ is a Walras equilibrium with fiat money (cf Florig [10]). Moreover, if for a.e. $t \in \mathcal{I}$, $x^*(t) \in \operatorname{cl} P_{i(t)}(x^*(t))$, then standard arguments imply that $\mathrm{q}_1 = 0$, this concluding the proof. □

Remark 4.1. *Since every Walras equilibrium is a weak equilibrium, Proposition 4.2 establish that under suitable conditions every Walras equilibrium of a convex economy could be approximated by a weak equilibria sequence and therefore by a rationing equilibria sequence.*

4.3 The general case

We now replace assumption **SA** by a more realistic one, assuming that every consumer could decide not to exchange anything (condition **S**, weak survival assumption). We will not assume however that he could survive for very long without exchanging anything. In such a case the limit of a sequence of rationing equilibria will not necessarily be a Walras equilibrium; indeed, it will be a hierarchic equilibrium, which is a competitive equilibrium with a segmentation of individuals according to their level of wealth. When this segmentation consists of just one group, then the hierarchic equilibrium reduces to a Walras equilibrium.

The next theorem, a generalization of Proposition 4.2, is the main result of this paper. The proof is given in the Appendix section.

Theorem 4.1. *Suppose $\mathcal{E}$ satisfies Assumptions* **C**, **P**, **M** *and* **S**, *and let $\{\mathcal{E}^n\}_{n\in\mathbb{N}}$ be a sequence of economies that approximates $\mathcal{E}$ and satisfying Assumptions* **A** *and* **F**. *For each $n \in \mathbb{N}$, let (x_n, y_n, p_n, q_n) be a weak equilibrium of $\mathcal{E}^n$, with $q_n > 0$ and $\| (p_n, q_n) \| = 1$. Then, there exists a hierarchic equilibrium $(x^*, y^*, \mathcal{P}, \mathcal{Q})$ for economy $\mathcal{E}$, with $\mathcal{P} = [\mathrm{p}_1, \dots, \mathrm{p}_k]^{\mathrm{t}}$, $\mathcal{Q} = (\mathrm{q}_1, \dots, \mathrm{q}_k)^{\mathrm{t}}$, $k \in \{1, \dots, L\}$, such that for some $\mathbf{N} \in \mathbb{N}^*_\infty$ the following hold:*

(i) for each $n \in \mathbf{N}$, $p_n = \sum_{r=1}^{k} \varepsilon_r(n)\, \mathrm{p}_r$, with $\varepsilon_{r+1}(n)/\varepsilon_r(n) \to_{\mathbf{N}} 0$,

(ii) for a.e. $t \in \mathcal{I}$, $x^*(t) \in \mathrm{acc}\{x_n(t)\}_{n\in\mathbf{N}}$, and for a.e. $t \in \mathcal{J}$, $y^*(t) \in \mathrm{acc}\{y_n(t)\}_{n\in\mathbf{N}}$.

Remark 4.2. *Since a rationing equilibrium is a weak equilibrium (see Definition 3.1), it follows that Theorem 4.1 remains valid when using a sequence of rationing equilibria instead of a sequence of weak equilibria as stated.*

5 Appendix: the proofs

The proof of Theorem 4.1 requires some additional definitions and technical results, presented in §5.1. The proof of Theorem 4.1 is given in §5.2.

5.1 Preliminary results

Both Definition 5.1 and Lemma 5.1 below are taken borrowed from Florig and Rivera [13].

Definition 5.1. *For integer $k \in \{1, \dots, m\}$, a set of orthonormal vectors $\{\psi_1, \dots, \psi_k\} \subset \mathbb{R}^m$ coupled with sequences $\varepsilon_r : \mathbb{N} \to \mathbb{R}_{++}$, $r \in \{1, \dots, k\}$, is called a lexicographic decomposition of a sequence $\psi : \mathbb{N} \to \mathbb{R}^m$, if there exists $\mathbf{N} \in \mathbb{N}^*_\infty$ such that following hold:*

(a) for all $r \in \{1, \dots, k-1\}$, $\varepsilon_{r+1}(n)/\varepsilon_r(n) \to_{\mathbf{N}} 0$,

(b) for all $n \in \mathbf{N}$, $\psi(n) = \sum_{r=1}^{k} \varepsilon_r(n)\psi_r$.

The lexicographic decomposition of $\psi : \mathbb{N} \to \mathbb{R}^m$ is denoted as $\{\{\psi_r, \varepsilon_r\}_{r=1}^{k}, \mathbf{N}\}$.

Lemma 5.1. *Every sequence $\psi : \mathbb{N} \to \mathbb{R}^m \setminus \{0_m\}$ admits a lexicographic decomposition.*

For the lexicographic decomposition above and $1 \leq r \leq k$, we set

$$\Psi(r) = [\psi_1, \dots, \psi_r] \in \mathbb{R}^{m\times r},$$

and for $z \in \mathbb{R}^m$ and $Z \subseteq \mathbb{R}^m$ we denote

$$\Psi(r)z = (\psi_1 \cdot z, \ldots, \psi_r \cdot z)^{\mathrm{t}} \in \mathbb{R}^r \quad \text{and} \quad \Psi(r)Z = \{\Psi(r)z : z \in Z\} \subseteq \mathbb{R}^r.$$

The next lemmata refers to a sequence $\psi : \mathbb{N} \to \mathbb{R}^m \setminus \{0_m\}$, whose lexicographic decomposition is $\{\{\psi_r, \varepsilon_r\}_{r=1}^k, \mathbf{N}\}$. Parts (i) and (ii) of this result are proved in Florig and Rivera [13], while part (iii) is a direct consequence of part (ii) coupled with the observation that for any $\xi \in \mathbb{R}^m$ and finite set of points $Z \subset \mathbb{R}^m$, $\operatorname{conv} \operatorname{argmax} \xi \cdot Z = \operatorname{argmax} \xi \cdot \operatorname{conv} Z$.

Lemma 5.2.

(i) For all $z \in \mathbb{R}^m$ there exists $\bar{n} \in \mathbb{N}$ such that for all $n > \bar{n}$ with $n \in \mathbf{N}$:

$$\Psi(k)\, z \leq_{lex} 0_k \quad \Longleftrightarrow \quad \psi(n) \cdot z \leq 0.$$

(ii) If $Z \subseteq \mathbb{R}^m$ is a finite set, then there exists $\bar{n} \in \mathbb{N}$ such that for all $n > \bar{n}$ with $n \in \mathbf{N}$:

$$\operatorname{argmax}_{lex} \Psi(k)\, Z \;=\; \operatorname{argmax}\, \psi(n) \cdot Z.$$

(iii) If $Z \subseteq \mathbb{R}^m$ is a convex and compact polyhedron, then there exists $\bar{n} \in \mathbb{N}$ such that for all $n > \bar{n}$ with $n \in \mathbf{N}$:

$$\operatorname{argmax}_{lex} \Psi(k)\, Z \;=\; \operatorname{argmax}\, \psi(n) \cdot Z.$$

Notice that both parts (ii) and (iii) in Lemma 5.2 remain valid when replacing $\operatorname{argmax}_{lex}$ by $\operatorname{argmin}_{lex}$.

The next lemmata is a key property used in the proof of our main result.

Lemma 5.3. *Let $Z \subset \mathbb{R}^m$ be a convex and compact polyhedron, and define*

$$\rho = \max\left\{r \in \{0, \ldots, k\} : \min_{lex} \Psi(r)Z = 0_{\max\{1,r\}}\right\} \quad \text{and} \quad \mathcal{F} = \operatorname{argmin}_{lex} \Psi(\rho)Z.$$

The following holds:

(i) $\limsup_{n\to\infty} \{z \in Z : \psi(n) \cdot z \leq 0\} \subseteq \operatorname{cl}\{z \in Z : \Psi(k)z \leq_{lex} 0_k\}.$

Suppose now that $\min_{lex} \Psi(k)Z <_{lex} 0_k$, and let $\{Z_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^m$ such that

$$\lim_{n\to\infty} Z_n = Z \quad \text{and} \quad \lim_{n\to\infty} (Z_n \cap \mathcal{F}) = Z \cap \mathcal{F}.$$

Then the following holds:

(ii) $\operatorname{cl}\{z \in Z : \Psi(k)z \leq_{lex} 0_k\} \subset \liminf_{n\to\infty} \{z \in Z_n \cap \mathcal{F} : \psi(n) \cdot z < 0\}.$

Proof. For $\mathbf{N} \in \mathbb{N}^*_\infty$ and $n' \in \mathbb{N}$, we set $\mathbf{N}_{n'} = \{n \in \mathbf{N} : n > n'\}$.

For the proof of part (i), let $\bar{z} \in \operatorname{argmin}_{lex} \Psi(k)Z$ and assume that $\limsup_{n\to\infty}\{z \in Z : \psi(n) \cdot z \leq 0\} \neq \emptyset$, since otherwise the result is trivial. Hence, for z^* in that subset, there is $\overline{\mathbf{N}} \in \mathbb{N}^*_\infty$ and $\{z_n\}_{n\in\overline{\mathbf{N}}} \subset Z$ such that $z_n \to_{\overline{\mathbf{N}}} z^*$ and for all $n \in \overline{\mathbf{N}}$, $\psi(n)\cdot z_n \leq 0$. By Lemma 5.2, part (iii), there exists $n_1 \in \mathbb{N}$ such that for all $n \in \overline{\mathbf{N}}_{n_1}$, we have

$$\operatorname{argmin}_{lex} \Psi(k)\, Z = \operatorname{argmin} \psi(n) \cdot Z.$$

As for all $n \in \overline{\mathbf{N}}_{n_1}$,

$$\psi(n) \cdot \bar{z} = \min \psi(n) \cdot Z \leq \psi(n) \cdot z_n \leq 0,$$

we have by part (i) of Lemma 5.2 that $\Psi(k)\,\bar{z} \leq_{lex} 0_k$.

Let $\sigma = \max\left\{r \in \{0,\ldots,k\} : \Psi(r)z^* = 0_{\max\{1,r\}}\right\}$. If $\Psi(k)z^* \leq_{lex} 0_k$, then the conclusion is trivial. Therefore, we assume $\Psi(k)z^* >_{lex} 0_k$, which implies that $\sigma < k$ and $\psi_{\sigma+1} \cdot z^* = \delta > 0$. At this stage, two cases must be considered.

Case 1. $\rho < \sigma$.

As $\rho < \sigma$, we have $\rho < k$, $\Psi(\rho+1)\,\bar{z} <_{lex} 0_{\rho+1}$ and $\Psi(\rho+1)\,z^* = 0_{\rho+1}$. Therefore, for all $\mu \in [0,1[$,

$$\Psi(\rho+1)\,(\mu\bar{z} + (1-\mu)z^*) <_{lex} 0_{\rho+1}.$$

Hence $\Psi(k)\,(\mu\bar{z} + (1-\mu)z^*) <_{lex} 0_k$, implying that $z^* \in \operatorname{cl}\{z \in Z \,:\, \Psi(k)z \leq_{lex} 0_k\}$.

Case 2. $\rho \geq \sigma$.

As $\rho \geq \sigma$, for all $r \in \{1,\ldots,\sigma\}$, $\psi_r \cdot \bar{z} = \psi_r \cdot z^* = 0$. Then $\{\bar{z}, z^*\} \subseteq \operatorname{argmin}_{lex} \Psi(\sigma)Z$. For $n \in \overline{\mathbf{N}}$ we set

$$\psi^*(n) = \sum_{r=1}^{\sigma} \varepsilon_r(n)\psi_r,$$

with $\psi^*(n) = 0$ when $\sigma = 0$. By part (ii) in Lemma 5.2 there exists $n_2 > n_1$ such that for all $n \in \overline{\mathbf{N}}_{n_2}$, $0 = \psi^*(n)\cdot\bar{z} = \psi^*(n)\cdot z^* \leq \psi^*(n)\cdot z_n$. For $n \in \overline{\mathbf{N}}$, we set

$$a_n = \sum_{r=1}^{\sigma+1} \varepsilon_r(n)\,\psi_r \cdot z_n \quad \text{and} \quad b_n = \frac{\varepsilon_{\sigma+2}(n)}{\varepsilon_{\sigma+1}(n)} \sum_{r=\sigma+2}^{k} \frac{\varepsilon_r(n)}{\varepsilon_{\sigma+2}(n)}\,\psi_r \cdot z_n,$$

with $b_n = 0$ if $\sigma + 1 = k$. By the fact that $\{z_n\}_{n\in\overline{\mathbf{N}}}$ remains in a compact set, there exists $n_3 > n_2$ such that for all $n \in \overline{\mathbf{N}}_{n_3}$, on the one hand

$$a_n \geq \varepsilon_{\sigma+1}(n)\,\psi_{\sigma+1}\cdot z_n > \varepsilon_{\sigma+1}(n)\frac{\delta}{2},$$

and, on the other hand, since b_n converges to zero,

$$b_n \in \frac{1}{4}[-\delta,\delta].$$

Therefore, for all $n \in \overline{\mathbf{N}}_{n_3}$,

$$0 \geq \psi(n) \cdot z_n = a_n + \varepsilon_{\sigma+1}(n) b_n \geq \varepsilon_{\sigma+1}(n) \frac{\delta}{4},$$

contradicting $\delta > 0$, hence concluding the proof of part (i).

In order to prove the part (ii), let $\bar{z} \in \operatorname{argmin}_{lex} \Psi(k)Z$. By the fact that $\min_{lex} \Psi(k)Z <_{lex} 0_k$, we have $\rho < k$ and $\psi_{\rho+1} \cdot \bar{z} < 0$. Let $\zeta \in \operatorname{cl}\{z \in Z : \Psi(k)z \leq_{lex} 0_k\}$. Then, for $\varepsilon \in]0,1]$ there exists $\zeta_\varepsilon \in \mathbb{B}(\zeta, \varepsilon/2) \cap Z$ such that $\Psi(k)\zeta_\varepsilon \leq_{lex} 0_k$. By the convexity of Z, for $\mu \in]0, \varepsilon/2[$ it follows that

$$z_\varepsilon = (1-\mu)\zeta_\varepsilon + \mu \bar{z} \in Z \cap \mathbb{B}(\zeta, \varepsilon),$$

and then $\Psi(k)\bar{z} \leq_{lex} \Psi(k)z_\varepsilon \leq_{lex} \Psi(k)\zeta_\varepsilon \leq_{lex} 0_k$.

The definition of ρ implies $\Psi(\rho)\bar{z} = 0_{\max\{1,\rho\}}$ and therefore we have also

$$\Psi(\rho) z_\varepsilon = \Psi(\rho)\zeta_\varepsilon = 0_{\max\{1,\rho\}}.$$

This last result coupled with the fact that $\rho < k$ implies

$$\psi_{\rho+1} \cdot \bar{z} \leq \psi_{\rho+1} \cdot z_\varepsilon \leq \psi_{\rho+1} \cdot \zeta_\varepsilon \leq 0 \quad \text{and} \quad \psi_{\rho+1} \cdot \bar{z} < 0.$$

Since $\psi_{\rho+1} \cdot \zeta_\varepsilon \leq 0$, we also have $\delta = \psi_{\rho+1} \cdot z_\varepsilon < 0$. Therefore $\Psi(\rho+1) z_\varepsilon <_{lex} 0_{\rho+1}$. Hence, we have established that $\Psi(k) z_\varepsilon <_{lex} 0$, $z_\varepsilon \in \mathcal{F}$ and $z_\varepsilon \in \mathbb{B}(\zeta, \varepsilon)$. Let us now consider $\{z_n\}_{n \in \mathbb{N}} \subseteq \mathcal{F} \cap Z_n$ with $z_n \to_{\mathbb{N}} z_\varepsilon$. We observe that

$$\psi(n) \cdot z_n = \sum_{r=1}^{k} \varepsilon_r(n) \psi_r \cdot z_n = \varepsilon_{\rho+1}(n) \, (\alpha_n + \beta_n),$$

where

$$\alpha_n = \frac{1}{\varepsilon_{\rho+1}(n)} \sum_{r=1}^{\rho+1} \varepsilon_r(n) \psi_r \cdot z_n \quad \text{and} \quad \beta_n = \frac{1}{\varepsilon_{\rho+1}} \sum_{r=\rho+2}^{k} \varepsilon_r(n) \psi_r \cdot z_n,$$

with $\beta_n = 0$ if $\rho + 1 = k$. Given that, for all $n \in \mathbb{N}$, $\Psi(\rho) z_n = 0_{\max\{1,\rho\}}$, and as β_n converges to 0 and $\delta < 0$, there exists $\bar{n}$ such that for all $n \in \mathbb{N}$ with $n > \bar{n}$, $\alpha_n < \delta/2$ and $\beta_n < -\delta/4$ and therefore $\alpha_n + \beta_n < \delta/4 < 0$. All of this implies that for all $n \in \mathbb{N}$ with $n > \bar{n}$,

$$\psi(n) \cdot z_n = \varepsilon_{\rho+1}(n) \, (\alpha_n + \beta_n) < \varepsilon_{\rho+1}(n) \, \delta/4 < 0.$$

Therefore, for $\zeta \in \operatorname{cl}\{z \in Z : \Psi(k)z \leq_{lex} 0_k\}$ and $\varepsilon \in]0,1]$, we have that

$$z_\varepsilon \in \mathbb{B}(\zeta, \varepsilon) \cap \liminf_{n \to \infty} \{z \in Z_n \cap \mathcal{F} : \psi(n) \cdot z < 0\},$$

and then, since the lim inf above is a closed set,[16] $\zeta \in \liminf_{n\to\infty}\{z \in Z_n \cap \mathcal{F} \,:\, \psi(n)\cdot z < 0\}$. □

5.2 Proof of Theorem 4.1

Proof. In the following, we use a sequence $(x_n, y_n, p_n, q_n)_{n\in\mathbb{N}}$ of weak equilibria with $q_n > 0$ of the economy $\mathcal{E}^n$ (which exists by Proposition 4.1, considering that a rationing equilibrium is a weak equilibrium). We can assume without loss of generality that for all $t \in \mathcal{I}$, $x_n(t) \in D^n_t(p_n, q_n)$ and all $t \in \mathcal{J}$, $y_n(t) \in S^n_j(p_n)$.[17]

In the remaining we split the proof of theorem into six steps.

Step 1 . *Hierarchic price.*

Since $\|(p_n, q_n)\| = 1$, $n \in \mathbb{N}$, from Lemma 5.1 there exist $\{\{(\mathrm{p}_r, \mathrm{q}_r), \varepsilon_r\}_{r=1,\ldots,k}, \mathbf{N}\}$, a lexicographic decomposition of the sequence $\{\psi(n) = (p_n, q_n)\}_{n\in\mathbb{N}}$. In the sequel, without loss of generality, we identify that subset $\mathbf{N}$ with $\mathbb{N}$, and we denote

$$\mathcal{P} = [\mathrm{p}_1, \ldots, \mathrm{p}_k]^{\mathrm{t}} \quad \text{and} \quad \mathcal{Q} = (\mathrm{q}_1, \ldots, \mathrm{q}_k)^{\mathrm{t}},$$

and for $r \in \{1, \ldots, k\}$, we set $\mathcal{P}(r) = [\mathrm{p}_1, \ldots, \mathrm{p}_r]^{\mathrm{t}}$ and $\mathcal{Q}(r) = (\mathrm{q}_1, \ldots, \mathrm{q}_r)^{\mathrm{t}}$.

Step 2. *Supply: for all $t \in \mathcal{J}$, $\limsup_{n\to\infty} S^n_{j(t)}(p_n) \subseteq S_{j(t)}(\mathcal{P})$.*

As for all $j \in J$, by Lemma 5.2 there exists $n_j \in \mathbb{N}$ such that for all $n > n_j$,

$$S_j(p_n) = S_j(\mathcal{P}) = \mathrm{argmax}_{lex}\mathcal{P}Y_j.$$

For all $n \in \mathbb{N}$ and all $j \in J$, $\mathrm{conv}Y^n_j = Y_j$, $S^n_j(p_n) \subseteq S_j(p_n) = \mathrm{conv}S^n_j(p_n)$, implying that for all $n > n_J = \max\{n_j,\, j = 1, \ldots, J\}$, and all $t \in \mathcal{J}$,

$$S^n_{j(t)}(p_n) \subseteq S_{j(t)}(\mathcal{P}) = \mathrm{conv}\, S^n_{j(t)}(p_n),$$

hence concluding the proof of this Step.

Step 3. *Income.*

For the sequel, for all $j \in J$, let $\zeta_j \in \mathrm{argmax}_{lex}\mathcal{P}Y_j$, and for all $i \in I$, we set $z_i = e_i + \sum_{j\in J}\theta_{ij}\lambda(T_j)\zeta_j$. By Step 2, for all $t \in \mathcal{I}$ and all $n > n_J$, $w_t(p_n, q_n) = p_n \cdot z_{i(t)} + q_n m(t)$.

Step 4. *Budget: For all $t \in \mathcal{I}$, $\limsup_{n\to\infty} B_t(p_n, q_n) \subseteq B_t(\mathcal{P}, \mathcal{Q})$. Moreover, if $m(t) > 0$ then*

$$B_t(\mathcal{P}, \mathcal{Q}) \quad\subseteq\quad \liminf_{n\to\infty}\left\{x \in X^n_{i(t)} \,:\, p_n \cdot x < w_t(p_n, q_n)\right\}.$$

[16] See, for example, Proposition 4.4 in Rockafellar and Wets [28].

[17] Since a countable union of negligible sets is negligible, we could restrict the sequel to an appropriate subset of full Lebesgue measure. Here, as the consumption and production sets are finite for each $n \in \mathbb{N}$, we could also adjust the sequence (x_n, y_n) such that for all $t \in \mathcal{I}$, $x_n(t) \in D^n_t(p_n, q_n)$ and all $t \in \mathcal{J}$, $y_n(t) \in S^n_j(p_n)$ while maintaining $(x_n, y_n) \in A(\mathcal{E}^n)$.

Using z_i from Step 3, the first inclusion is a straightforward consequence of part (i) of Lemma 5.3 applied to $Z = \big(X_{i(t)} - z_i\big) \times \{-m(t)\}$. Indeed, note that for all $n \in \mathbb{N}$, $n > n_J$, and all $x'_n(t) \in B_t(p_n, q_n)$ we have $\psi_n \cdot z_n \leq 0$ with $z_n = (x'_n(t) - z_{i(t)}, -m(t))$ and $\psi(n) = (p_n, q_n)$.

For the second inclusion, for $t \in \mathcal{I}$ and $n \in \mathbb{N}$, we set $\mathcal{F} = \min_{lex} \mathcal{P}(\rho) X_{i(t)}$ and

$$\rho = \max\left\{r \in \{0, \dots, k\} \,:\, \min_{lex} \mathcal{P}(r) X_{i(t)} = 0_{\max\{1,r\}}\right\}.$$

Assumption **S** coupled with the observation $m(t)\mathcal{Q} >_{lex} 0_k$ implies

$$\min_{lex} \mathcal{P}(\big(X_{i(t)} - z_{i(t)}\big) - m(t)\mathcal{Q} <_{lex} 0_k \quad \text{and} \quad m(t)\mathcal{Q}(\rho) = 0_{\max\{1,\rho\}}.$$

Therefore, producers profit maximization and Assumption **S** implies

$$\left(\{e_{i(t)}\} + \sum_{j \in J} \theta_{i(t)j} \lambda(T_j)\, Y_j\right) \cap X_{i(t)} \subseteq \mathcal{F}.$$

By part (iii) of Lemma 5.2 we observe that $\mathcal{F}$ is a face of $X_{i(t)}$, and then, by Assumption **F** it follows that

$$\lim_{n \to \infty} X^n_{i(t)} \cap \mathcal{F} = X_{i(t)} \cap \mathcal{F}.$$

By part (ii) of Lemma 5.3

$$B_t(\mathcal{P}, \mathcal{Q}) \quad \subseteq \quad \liminf_{n \to \infty} \left\{x \in X^n_{i(t)} \cap \mathcal{F} \,:\, p_n \cdot (x - z_{i(t)}) < q_n m(t)\right\},$$

and since

$$\liminf_{n \to \infty} \left\{x \in X^n_{i(t)} \cap \mathcal{F} \,:\, p_n \cdot (x - z_{i(t)}) < q_n m(t)\right\} \quad \subseteq \quad \liminf_{n \to \infty} \left\{x \in X^n_{i(t)} \,:\, p_n \cdot x < w_t(p_n, q_n)\right\}$$

the second inclusion holds true.

Step 5. *Demand: for all $t \in \mathcal{I}$ with $m(t) > 0$ and all $x^*(t) \in \operatorname{acc}\{x_n(t)\}_{n \in \mathbb{N}}$,*

$$P_{i(t)}(x^*(t)) \cap B_t(\mathcal{P}, \mathcal{Q}) = \emptyset.$$

Let $t \in \mathcal{I}$ such that $m(t) > 0$ and choose $\mathbf{N}(t) \in \mathbb{N}^*_\infty$ such that $x_n(t) \to_{\mathbf{N}(t)} x^*(t)$ and for all $n \in \mathbf{N}(t)$, $n > n_J$. By contraposition, assume that there is $\xi \in P_{i(t)}(x^*(t)) \cap B_t(\mathcal{P}, \mathcal{Q})$. Then, by Step 5 there exists $\bar{n}_1 > n_J$ and $\xi_n \to_{\mathbb{N}} \xi$ such that for all $n > \bar{n}_1$ with $n \in \mathbb{N}$,

$$p_n \cdot (\xi_n - z_{i(t)}) - q_n m(t) < 0 \quad \text{and} \quad \xi_n \in X^n_{i(t)}.$$

As the graph of $P_{i(t)}$ is open, there exists $\bar{n}_2 > \bar{n}_1$ such that for all $n > \bar{n}_2$ with $n \in \mathbb{N}$,

$$p_n \cdot (\xi_n - z_{i(t)}) - q_n m(t) < 0 \quad \text{and} \quad \xi_n \in X^n_{i(t)} \cap P_{i(t)}(x^*(t)),$$

and, again, as the graph of $P_{i(t)}$ is open, we can choose $\bar{n}_3 > \bar{n}_2$ such that for all $n > \bar{n}_3$ with $n \in N(t)$, we have $p_n \cdot (\xi_n - z_{i(t)}) - q_n m(t) < 0$ and $\xi_n \in P^n_{i(t)}(x_n(t))$. As $q_n m(t) > 0$, the last fact contradicts $x_n(t) \in D^n_t(p_n, q_n)$ for all $n > \bar{n}_3$ with $n \in N(t)$ (see Proposition 3.1).

Step 6. *Equilibrium allocation.*

Using Fatou's lemma in Artstein [1], there exists $(x^*, y^*) \in A(\mathcal{E})$ such that for a.e. $t \in \mathcal{I}$ and a.e. $t' \in \mathcal{J}$, $x^*(t) \in \text{acc}\{x_n(t)\}_{n\in\mathbb{N}}$ and $y^*(t') \in \text{acc}\{y_n(t')\}_{n\in\mathbb{N}}$. By Step 2, for a.e. $t \in \mathcal{J}$, $y^*(t) \in S_{j(t)}(\mathcal{P})$, and by Steps 4 and 5, for a.e. $t \in \mathcal{I}$, $x^*(t) \in B_t(\mathcal{P}, \mathcal{Q})$ and $P_{i(t)}(x^*(t)) \cap B_t(\mathcal{P}, \mathcal{Q}) = \emptyset$. $\square$

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