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Hierarchic competitive equilibria☆

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Abstract

Without an interiority or strong survival assumption, an equilibrium may not exist in the standard Arrow–Debreu model. We propose a generalized concept of competitive equilibrium, called hierarchic equilibrium. Instead of using standard prices we use hierarchic prices. Existence will be shown without a strong survival assumption and without a non-satiation condition on the preferences. Under standard assumptions this reduces to the Walras equilibrium. Hierarchic equilibria are weakly Pareto optimal and any Pareto optimum can be decentralized without a border condition. We prove the existence of a Pareto optimal hierarchic equilibrium under additional assumptions. Later, we establish a core equivalence result. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

In the Arrow–Debreu model, a Walras equilibrium may not exist without a strong survival assumption. Several authors Gay (1978), Danilov and Sotskov (1990), Marakulin (1990), and Mertens (1996) advance generalized equilibrium concepts which exist without a strong survival assumption. The scope of these concepts is however restricted to exchange economies with a particular type of consumption sets. The purpose of this paper is to propose and prove the existence of a generalized equilibrium concept for economies with production and convex consumption sets.

The possibility of minimum-wealth situations at some prices may lead to non-existence of a Walras equilibrium. Usually one prevents this situation from arising by an interiority

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condition, also called strong survival or Slater assumption. Roughly, such a condition asserts that every consumer can consume some bundle of goods within the interior of his consumption set without exchanging anything (cf. Arrow and Debreu, 1954). This is satisfied, for example, if every consumer's initial endowment is in the interior of his consumption set and if each firm can remain inactive. In the case where the consumption sets correspond to the positive orthant, the strong survival assumption means that every consumer is initially endowed with a strictly positive quantity of every existing commodity. Most consumers have however a single commodity to sell their labor. So one might argue that the strong survival assumption is almost never satisfied. Furthermore, should an agent have no share in any of the firms, then this condition can only hold if the consumption set has a non-empty interior. This means that every commodity — so, even every input or industrial by-product — must be consumable.

A weak survival assumption (also called autarky assumption) asserts that every consumer can consume some bundle of goods in his consumption set without exchanging anything. So consumers might have a consumption set with empty interior and their initial endowment may lie on the border of their consumption set.

A weak survival assumption is by far more acceptable than a strong one. Several authors established sufficient existence conditions which are stronger than the weak survival assumption, but weaker than the strong survival assumption. Considering instead of the interiority condition, some sort of connection between the agents via their preferences and initial endowments, Gale (1957, 1976) established the existence of a Walras equilibrium for linear exchange economies. He assumed that every consumer has some commodity some other consumer likes, and there are no two subgroups such that group 1 has commodities group 2 likes, but group 2 has no commodities group 1 likes. Furthermore, the weak survival assumption must hold. One calls then the economy irreducible. Several authors adapted this idea to more standard economies establishing weaker conditions for the existence of a Walras equilibrium than those stated in Arrow and Debreu (1954). A list of such contributions includes McKenzie (1959, 1961, 1981), Debreu (1962), Arrow and Hahn (1971), Moore (1975), Bergstrom (1976), Spivak (1978), Florenzano (1981), Geistdoerfer-Florenzano (1982), Hammond (1993), Maxfield (1997), and Florig (2001).

Despite all these efforts to find weak conditions replacing the strong survival assumption, there are extremely simple and economically meaningful examples for the non-existence of a Walras equilibrium (cf. Gale, 1976). The question then arises, if a competitive economy can be in some kind of equilibrium situation, even if a Walras equilibrium does not exist. If so, it would be interesting whether analogs of the Welfare theorems and core equivalence still hold.

Partial answers have been given in the literature (Gay, 1978; Danilov and Sotskov, 1990; Marakulin, 1990). We try to investigate these issues. First, we give in Section 2 a definition of a competitive equilibrium concept we call hierarchic equilibrium. Walras and dividend equilibria are special cases of the hierarchic equilibrium. A hierarchic equilibrium is a kind of dividend equilibrium using hierarchic prices which generalize the notion of exchange rates introduced by Gay (1978). At an exchange rate the set of commodities is split into several sub-markets according to their price level. A commodity of sub-market 2 cannot buy commodities of sub-market 1, it buys other commodities of sub-market 2 at a strictly positive price and it buys commodities of sub-markets 2,3,... at price zero, so it buys infinite amounts of these goods. In Section 3, we first study an easy special case of hierarchic equilibria. Then we give an interpretation of hierarchic equilibria in terms of dividend equilibria of economies with small indivisibilities. We illustrate this with various examples. In Section 4, we introduce further notations and a technical lemma. In Section 5, we state the assumptions. In Section 6, we give the existence proof. In Section 7, we show that under standard assumptions, ensuring the existence of a Walras equilibrium, the hierarchic equilibrium reduces to a Walras equilibrium. In Section 8, we show that hierarchic equilibria are weakly Pareto optimal, we give an analog of the second Welfare theorem without the usual border condition and we prove the existence of a Pareto optimal hierarchic equilibrium under additional assumptions. In Section 9, we give a core equivalence result. In Section 10, we show that one may impose a monotonicity property, implying equal treatment. In Section 11, we discuss the different generalized equilibrium concepts and their links to the hierarchic equilibrium.

2. Model

We denote by N, Z, R, respectively, the set of natural, integer and real numbers. Let $I = \{1, \dots, I\}$, $J = \{1, \dots, J\}$ and $L = \{1, \dots, L\}$ be finite sets of consumers, firms and commodities. Each firm $j \in J$ is characterized by a production set $Y_j \subset R^L$. We denote the aggregate production set by $Y = \sum_{j\in J} Y_j$. Every consumer $i \in I$ is characterized by his consumption set $X_i \subset R^L$, his initial endowment $\omega_i \in R^L$ and his preference correspondence $P_i : \prod_{i\in I} X_i \times \prod_{j\in J} Y_j \to 2^{X_i}$. Let $\omega = \sum_{i\in I}\omega_i$ be the total initial endowment vector, and let $X = \sum_{i\in I} X_i$. For all $(i, j) \in I \times J$, $\theta_{ij} \in [0, 1]$ represents consumer i's share in firm j. For every $j \in J$, $\sum_{i\in I}\theta_{ij} = 1$.

An economy $\mathcal{E}$ is a collection:

$$\mathcal{E} = ((X_i, P_i, \omega_i)_{i\in I}, (Y_j)_{j\in J}, (\theta_{ij})_{(ij)\in I\times J}).$$

We will denote the set of feasible consumption–production plans by

$$F(\mathcal{E}) = \left\{ (x, y) \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j \,\middle|\, \sum_{i\in I} x_i = \sum_{j\in J} y_j + \omega \right\}.$$

Let $\bar{R} = (R \cup \{+\infty\})$. For any $n \in N$, let $\geq$ be the lexicographic order [1] on $\bar{R}^n$. Extrema will be taken with respect to the lexicographic order. We adopt the convention $0(+\infty) = 0$. For $x \in R^n$, note $\operatorname{supp} x = \{h \in \{1, \dots, n\} | x_h \neq 0\}$ the support of x. Note $e_1, \dots, e_n$ the canonic basis and $\mathbb{B} = \{x \in R^n | \|x\| \leq 1\}$ the closed unit ball in R^n.

Definition 1. A *hierarchic price* is a finite ordered family $\mathcal{P} = \{p^1, \dots, p^k\}$ of vectors of R^L.

[1] For $(s, t) \in \bar{R}^n \times \bar{R}^n$, $s \geq t$, if $s_r < t_r$, $r \in \{1, \dots, n\}$ implies that $\exists \rho \in \{1, \dots, r-1\}$ such that $s_\rho > t_\rho$. We write $s > t$ if $s \geq t$, but not $[t \geq s]$.

If $k = 1$, this reduces to the standard case. We denote by $\mathcal{HP}$ the set of hierarchic prices. The number k is determined at the equilibrium. We will see that one never needs k to be greater than L.[2]

For $\mathcal{P} \in \mathcal{HP}$ and $x \in R^L$, we define the *value* of x to be

$$\mathcal{P}x = (p^1 \cdot x, \dots, p^k \cdot x) \in \bar{R}^k.$$

The *supply* of firm $j \in J$ at the price $\mathcal{P}$ is

$$S_j(\mathcal{P}) = \{y_j \in Y_j | \forall z_j \in Y_j,\ \mathcal{P}z_j \leq \mathcal{P}y_j\}.$$

Alternatively, we can write $S_j(\mathcal{P}) = \operatorname{argmax} p^k\{\operatorname{argmax} p^{k-1} \dots \{\operatorname{argmax} p^1 Y_j\} \dots\}$. Given a hierarchic price, firms are thus supposed to maximize the profit lexicographically. The *profit* of firm $j \in J$ is

$$\pi_j(\mathcal{P}) = \sup_{y_j \in Y_j} \mathcal{P}y_j.$$

A *hierarchic revenue* is a vector $w \in \bar{R}^k$. For all $i \in I$, all $\mathcal{P} \in \mathcal{HP}$, all $w \in \bar{R}^k$ let

$$r_i(\mathcal{P}, w) = \min\{r \in \{1, \dots, k\} | \exists x \in X_i, (p^1 \cdot x, \dots, p^r \cdot x) < (w^1, \dots, w^r)\},$$
$$v_i(\mathcal{P}, w) = (w^1, \dots, w^{r_i(\mathcal{P}, w)}, +\infty, \dots, +\infty) \in \bar{R}^k.$$

The *budget set* of consumer i, with respect to $\mathcal{P} \in \mathcal{HP}$ and $w \in \bar{R}^k$ will be

$$B_i(\mathcal{P}, w) = \{x_i \in X_i | \mathcal{P}x_i \leq v_i(\mathcal{P}, w)\}.$$

Lemma 1. *If X_i is closed convex, then $B_i(\mathcal{P}, w) = \overline{\{x_i \in X_i | \mathcal{P}x_i \leq w\}}$.*

Proof. Note first that if X_i is closed, then $\overline{\{x_i \in X_i | \mathcal{P}x_i \leq w\}} \subset B_i(\mathcal{P}, w)$. For the converse, it is sufficient to note that for every couple $(a, b) \in \{x_i \in X_i | \mathcal{P}x_i < w\} \times B_i(\mathcal{P}, w)$ and every $\lambda \in]0, 1]$, $\lambda a + (1 - \lambda)b \in \{x_i \in X_i | \mathcal{P}x_i < w\}$ by the convexity. Then, $b \in \overline{\{x_i \in X_i | \mathcal{P}x_i \leq w\}}$. □

Definition 2. A collection $(x, y, \mathcal{P}, w) \in F(\mathcal{E}) \times \mathcal{HP} \times (\bar{R}^k)^I$ is a *hierarchic equilibrium* of the economy $\mathcal{E}$ if:[3]

1. for all $i \in I$, $x_i \in B_i(\mathcal{P}, w_i)$ and $P_i(x, y) \cap B_i(\mathcal{P}, w_i) = \emptyset$;
2. for all $i \in I$, $\mathcal{P}\omega_i + \sum_{j \in J} \theta_{ij} \pi_j(\mathcal{P}) \leq w_i$;
3. for all $j \in J$, $y_j \in S_j(\mathcal{P})$.

[2] The forthcoming definitions will depend for any $r \in \{2, \dots, k\}$ only on the non-zero part of p^r which is orthogonal to $p^1, \dots, p^{r-1}$. Therefore, by an inductive argument we can always transform a hierarchic price into an equivalent one consisting of two by two orthogonal vectors (thus of at most L).

[3] If we note L_i the lineality space of the positive cone generated by consumer i's net trade set and d_i the codimension of L_i, then we may reduce any hierarchic price into an equivalent one with $k \leq 1 + \min_{i \in I} d_i$. Indeed, either 0 is an equilibrium price or we may assume the prices two by two orthogonal and all non-zero (cf. Footnote 2). The rank of consumer i is smaller or equal to the index of the first vector which is not orthogonal to L_i. The prices of a higher index are irrelevant to him.

Definition 3. A dividend equilibrium (resp. Walras equilibrium) is a hierarchic equilibrium $(x, y, \mathcal{P}, w) \in F(\mathcal{E}) \times \mathcal{HP} \times (\bar{R}^k)^I$ with $k=1$ (resp. $k=1$ and $\mathcal{P}\omega_i + \sum_{j\in J}\theta_{ij}\pi_j(\mathcal{P}) = w_i$).

Let $(x, y, \mathcal{P}, w)$ be a hierarchic equilibrium $(x, y, \mathcal{P}, w)$. If for all $i \in I$,

$$w_i^1 > \inf p^1 \cdot X_i,$$

then (x, y, p^1, w^1) is a dividend equilibrium. If furthermore, all consumers satisfy some non-satiation assumption, then (x, y, p^1) is a Walras equilibrium. Note that the condition excluding minimum-wealth situations holds under a global interiority condition together with an irreducibility condition (cf. Section 7). In particular, it is implied by a global non-satiation assumption together with the strong survival assumption:

$$0 \in \operatorname{int}\left(X_i - \omega_i - \sum_{j\in J}\theta_{ij}Y_j\right), \quad \text{for all } i \in I.$$

Thus, hierarchic equilibrium and the Walras equilibrium coincide under the standard assumptions for the existence of a Walras equilibrium. However, even if the set of Walras equilibria is non-empty, there may exist hierarchic equilibrium allocations not in the set of Walras equilibrium allocations.

Example 1. Consider an economy with two consumers and two commodities. Let $X_1 = X_2 = R_+^2$, $\omega_1 = (1, 1)$, $\omega_2 = (0, 1)$ and $u_1(x) = x^1$, $u_2(x) = \min\{x^2, 2\}$ are the utility functions. The unique Walras equilibrium is $x_1 = (1, 0)$, $x_2 = (0, 2)$, $p = (1, 0)$. Other hierarchic equilibria exist. For example, $((\xi_1, \xi_2), \{p^1, p^2\}, (w_1, w_2))$ with $\xi_i = \omega_i$, $i = 1, 2$, $p^1 = (1, 0)$, $p^2 = (0, 1)$ and $w_1 = (1, +\infty)$, $w_2 = (0, 1)$.

In an economy where the strong survival assumption holds, but where satiation points of the preferences may exist, dividends may be necessary in order to lead the economy to an equilibrium situation, called dividend equilibrium or competitive equilibrium with slack (Drèze and Müller, 1980; Makarov, 1981; Aumann and Drèze, 1986; Mas-Colell, 1992). At a hierarchic equilibrium some agents are restricted to operate in an affine subspace of the commodity space. This affine subspace is determined only at the equilibrium. Even if preferences are not satiated on the whole consumption set, they could be satiated on the intersection between the consumption set and an affine subspace, especially if this intersection is compact. We will see in the next section, in Example 4, that the dividend structure in the definition of the hierarchic equilibrium is necessary for the existence when considering consumption sets which differ from the positive orthant. We refer to Section 6 for an interpretation of the dividend structure and to Section 10 to see that a dividend structure should be considered, even if the existence can be shown without, as it is the case under restrictive assumptions.

3. Interpretation

Before giving an economic interpretation of the hierarchic equilibrium, we first study a special case which might be easier to understand in a first step.

Definition 4. An *exchange rate* is a hierarchic price $\mathcal{P} = \{p^1, \dots, p^k\} \subset R_+^L$ such that $r \neq r'$ implies $\mathrm{supp}(p^r) \cap \mathrm{supp}(p^{r'}) = \emptyset$.

Proposition 1. *Let $\mathcal{E}$ be an economy with $J = \emptyset$, for all $i \in I$, $X_i = R_+^L$, for all $x \in F(\mathcal{E})$, for all $\xi_i, \bar{\xi}_i \in R_+^L$, $\xi_i \in P_i(x)$ implies $\xi_i + \bar{\xi}_i \in P_i(x)$.* Let *$(x, w, \mathcal{P})$ be a hierarchic equilibrium of $\mathcal{E}$. Then, there exists an exchange rate $\mathcal{Q}$* and $(w_i') \in (\bar{R}^k)^I$ *such that $(x, \mathcal{Q}, w')$ is a hierarchic equilibrium.*

Proof. Let $\bar{r} + 1$ be the smallest element of $\{1, \dots, k\}$ such that for some $h \in L$, $(p_h^1, \dots, p_h^{\bar{r}+1}) < 0$. Then, for all $i \in I$, $r_i(\mathcal{P}, w_i) \leq \bar{r} + 1$. Set $\bar{P} = \{p^1, \dots, p^{\bar{r}}\}$ and $\bar{w}_i = (w_i^1, \dots w_i^{\bar{r}})$ for all $i \in I$. For all $y \in B_i(\bar{P}, \bar{w}_i)$ there exists $t > 0$ such that $y + te_h \in B_i(\bar{\mathcal{P}}, \bar{w}_i)$. If $y \in P_i(x)$ then $y + te_h \in P_i(x)$. Hence $(x, \bar{P}, \bar{w})$ is a hierarchic equilibrium.

For $h \in L$, choose the smallest $r \in \{1, \dots, \bar{r}\}$ such that $0 < p_h^r$. For all $\rho \in \{r + 1, \dots, \bar{r}\}$ replace p_h^ρ by 0. Note the new hierarchic price by $\bar{\mathcal{P}}_h$. Then, $(x, \bar{\mathcal{P}}_h, \bar{w})$ is a hierarchic equilibrium. Indeed, suppose without loss of generality that for all $i \in I$, $\bar{w}_i = v_i(\bar{\mathcal{P}}, \bar{w}_i)$. Either $\bar{w}_i^r > 0$, then $B_i(\bar{\mathcal{P}}, \bar{w}_i) = B_i(\{p^1, \dots, p^r\}, \bar{w}_i)$ or otherwise $\bar{w}_i^r = 0$, then $B_i(\bar{\mathcal{P}}, \bar{w}_i) \subset R^{h-1} \times \{0\} \times R^{L-h-1}$ and h cannot be consumed anyway. We may apply this L times iteratively obtaining $\mathcal{Q} \in \mathcal{HP} \cap R_+^L$ such that $0 < q_h^r$ for some $r \in \{1, \dots, \bar{r}\}$ and $h \in L$ implies $q_h^\rho = 0$ for all $\rho \in \{r+1, \dots, \bar{r}\}$ and such that $(x, Q, \bar{w})$ is a hierarchic equilibrium. So Q is an exchange rate. □

So in this simple setting one could think of a partition of the set of commodities into different sub-markets (cf. Gay, 1978; Danilov and Sotskov, 1990). The commodities are ordered by the level of their price. For example, second class commodities may cost infinity compared to third class commodities, but they may be worthless compared to first class commodities. Then, the owners of second class commodities obtain third class commodities for free, but first class commodities are inaccessible to them if they do not own some of them. So the partition on the commodity set establishes also a partition of the consumers into k types. Consumers of type r have strictly positive wealth in sub-market r, zero wealth in sub-markets $1, \dots, r-1$ and infinite wealth in sub-markets $r+1, \dots, k$. So commodities of the sub-markets $1, \dots, r-1$ are inaccessible to them and commodities in sub-markets $r+1, \dots, k$ are so cheap for them that the price of these is negligible in their budget constraint.

Under conditions which ensure the existence of a Walras equilibrium all consumers have nonminimal-wealth and they are all able to buy a strictly positive quantity of every commodity. So in this sense they have all an income of the same order. Such a situation arises if the initial endowments are in the interior of the consumption sets. So mathematically this is generic provided the consumption sets have non-empty interior.

However, all consumers having access to all commodities is a rather extreme case which is probably almost never satisfied. In most economies, agents have incomes of different orders in the sense that some commodities have a negligible price for some consumers, a non-negligible for others and an infinite price for still another group of consumers. An infinite price should of course be understood in the sense that even by spending his entire wealth on a commodity one could not buy a unit of it. All this is of course related to the fact that there are no perfectly divisible commodities as assumed in the standard model. Using R^L as a commodity space is of course only an approximation of the "real" discrete commodity space. The rational is that the commodities one considers are "almost perfectly" divisible in the sense that the indivisibilities are small and insignificant enough so that they can be neglected. Our point here is that in the absence of the strong survival assumption there are no insignificant indivisibilities. Clearly, at any level of indivisibility of the commodities one may find a price such that not all consumers have access to all commodities. Instead of working with a discrete commodity space, working with hierarchic prices, we can capture phenomena related to indivisibilities which may occur at any arbitrary small level of indivisibility of the commodities. Under a strong survival assumptions these phenomena cannot occur. Every consumer has a strictly positive quantity of everything and so every consumer has of course access to all commodities

So one should not think of hierarchic equilibria as economic situations with several price vectors, but one should rather see the hierarchic equilibrium as a dividend equilibrium of an economy with small indivisibilities. Suppose it is not possible to divide commodities beyond a certain level, say $(1/n) > \varepsilon$ for some large integer n. Think of a "true" market price as equal to $\sum_{r=1}^{k}\varepsilon^{r-1}p^r$, for some $\varepsilon > 0$ much smaller than $1/n$. In Florig (2000) it is proven that each hierarchic equilibrium of a linear exchange economy is the limit of dividend equilibria of the same economy where all commodities are supposed to be indivisible with a level of indivisibility going to zero. Conversely, Florig and Rivera (2001) propose a generalization of the Walras equilibrium existing in the case of discrete consumption sets. There standard prices are used and considering a sequence of economies with vanishing indivisibilities the corresponding equilibria converge to a hierarchic equilibrium.

Example 2. Let $X_1 = X_2 = R^2_+$, $\omega_1 = (1, 1)$, $\omega_2 = (0, 1)$ and the utility functions are $u_1(x) = x^1$ and $u_2(x) = x^1 + x^2$. Neither a Walras nor a dividend equilibrium exist. The hierarchic equilibria are $\mathcal{P} = \{p^1, p^2\}$ with $p^1 = (1, 0)$, $p^2 = (0, 1)$, $x_1 = (1, t)$, $x_2 = (0, 2 - t)$, $w_1 = (1, +\infty)$, $w_2 = (0, 2 - t)$ with $t \in [0, 1]$. Suppose we may divide any commodity into $n \in N$ minimal units. The new consumption sets are thus $X^n_1 = X^n_2 = \{x \in R^2_+ | nx \in Z^2_+\}$. Note that X^n_i converges to R^2_+ as n goes to infinity. The following is a dividend equilibrium of the corresponding discrete economy: $p^n = (1, (1/3n))$, $x^n_1 = (1, t^n)$, $x_2 = (0, 2 - t^n)$, $w^n_1 = 1 + (1/3n)$, $w^n_2 = (1/3n)(2 - t^n)$ with t^n being a rational in $[t, t + (1 - t)/n]$ such that $nt^n \in Z_+$.

So whatever the level of indivisibility, consumer 2 cannot access the market of good 1 at equilibrium. The hierarchic price enables us to approximate a discrete consumption space by R^2_+ capturing the phenomena observed at any level of indivisibility, that is, no one has enough of good 2 in order to use it for buying some of good 1; the price of good 2 is far from being small for consumer 2, but it is almost zero for consumer 1. Note also that in this example the worst individual rational feasible allocation, i.e. $x_i = \omega_i$ is an equilibrium allocation.

Example 3. Consider an exchange economy with three consumers and three commodities; for all $i \in I$, $X_i = R^3_+$, $u_1(x) = x^1 - x^2 - x^3$, $u_2(x) = x^1 + 2x^2 + x^3$, $u_3(x) =$

$x^1+x^2+2x^3, \omega_1=(1,1,1), \omega_2=\omega_3=(0,0,0)$. The hierarchic equilibrium allocations are $x_1=(1,0,0), x_2^t=(0,1,t), x_3^t=(0,0,1-t)$ for $t\in[0,1]$ with the hierarchic price $p^1=(1,0,0), p^2=(0,2,1)$ and $y_1=(1,0,0), y_2^t=(0,1-t,0), y_3^t=(0,t,1)$ for $t\in[0,1]$ with the hierarchic price $q^1=(1,0,0), q^2=(0,1,2)$. It is sufficient to take $p^n=p^1+(1/4n)p^2$ and $q^n=q^1+(1/4n)q^2$ in order to approach the respective hierarchic equilibria by dividend equilibria of the economy where each unit of a commodity may be divided into n minimal units, i.e. $X_i^n=\{x\in R_+^3 | nx\in Z_+^3\}$.

The limit of both price sequences is $p=(1,0,0)$. At this price one could exchange good 2 against good 3 at any exchange rate. So p is not a good approximation of the equilibrium price of a discrete economy. We would totally neglect at which rate one may exchange good 2 against good 3 and again goods 2 and 3 are far from being cheap for consumers 2 and 3, but they are for consumer 1.

Example 4. The following example may not be interpreted in terms of sub-markets. The interpretation in terms of indivisibilities of the hierarchic price remains nevertheless valid. Let $X_1=X_2=\{x\in R_+^2 | x^1+x^2\geq 3\}, \omega_1=(3,1), \omega_2=(2,1)$ and $u_1(x)=x^1, u_2(x)=x^2$. The set of Pareto optimal allocations and the core reduce to the singleton $x_1=(4,0), x_2=(1,2)$. Nevertheless, no Walras or dividend equilibrium exists. The only hierarchic equilibrium is $x_1=(4,0), x_2=(1,2), p^1=(1,1), p^2=(-1,1)$ and $w_1=(4,-2), w_2=(3,1)$. Indeed, we must have $p^1=(1,1)$ since otherwise we would have a Walras or dividend equilibrium. Consumer 1 is not at minimum-wealth with respect to p^1. Thus, his demand is $x_1=(4,0)$ if he gets no extra revenue at this level. Otherwise he would demand even more than 4 units of good 1 and this is not feasible. Then we must have $x_2=(1,2)$ and this is only possible with $p^2=(-1,1)$ and $w_1=(4,-2), w_2=(3,1)$.

For any discretization of the consumption set (as described above), there exists a real $\varepsilon>0$ such that $(p^1+\varepsilon p^2, x_1, x_2)$ is a dividend equilibrium. Taking a sequence of discretizations converging to the initial convex consumption set one obtains at the limit (in a certain sense) the hierarchic equilibrium.

4. Further notations

The notations in this section are needed for most of the proofs and for Section 7. For a set $Z\subset R^L$, we denote the convex hull of Z by

$$\mathrm{co}Z=\left\{\sum_{n=1}^{m}\lambda_n z_n | z_n\in Z,\ \lambda_n\geq 0,\ \sum_{n=1}^{m}\lambda_n=1, m\in N\right\},$$

the positive hull of Z by,

$$\mathrm{pos}Z=\left\{\sum_{n=1}^{m}\lambda_n z_n | z_n\in Z,\ \lambda_n\geq 0,\ m\in N\right\},$$

the linear span of Z by

$$\mathrm{span}Z = \mathrm{pos}Z - \mathrm{pos}Z,$$

the lineality of pos$Z$ by

$$LZ = -\mathrm{pos}Z \cap \mathrm{pos}Z,$$

and the orthogonal complement of Z by

$$Z^{\perp} = \{q \in R^L | q \cdot z = 0, \ \forall z \in Z\}.$$

Given an economy $\mathcal{E}$, we will often abuse of the notations writing pos$\mathcal{E}$, span$\mathcal{E}$, ... for, respectively, pos$(X - Y - \{\omega\})$, span$(X - Y - \{\omega\})$, ...

Given an economy $\mathcal{E}$ and a sequence of vectors $\{q^1, \dots, q^r\} \subset R^L$ let for all $i \in I$,

$$X_i(q^1, \dots, q^r) = \{x_i \in X_i(q^1, \dots, q^{r-1}) | q^r \cdot x_i = \inf q^r \cdot X_i(q^1, \dots, q^{r-1})\},$$

with $X_i(\emptyset) = X_i$ and for all $j \in J$, let

$$Y_j(q^1, \dots, q^r) = \{y_j \in Y_j(q^1, \dots, q^{r-1}) | q^r \cdot y_j = \sup q^r \cdot Y_j(q^1, \dots, q^{r-1})\},$$

with $Y_j(\emptyset) = Y_j$. We have a new economy

$$\mathcal{E}(q^1, \dots, q^r) = ((X_i(q^1, \dots, q^r), P_i, \omega_i)_{i \in I}, (Y_j(q^1, \dots, q^r))_{j \in J}, (\theta_{ij})_{(ij) \in I \times J}),$$

which is a restriction of the original economy. Typically, $\{q^1, \dots, q^r\}$ will be a basis of the orthogonal complement of $L(\mathcal{E})$ in R^L or in span$\mathcal{E}$.

For an economy $\mathcal{E}$, let

$$\mathcal{Z}(\mathcal{E}) = \{\sigma \in R^L | \sup \sigma \cdot (Y + \{\omega\}) \le \inf \sigma \cdot X\}.$$

Note that if $\sigma \in \mathcal{Z}(\mathcal{E})$, then $F(\mathcal{E}) = F(\mathcal{E}(\sigma))$. If for every $\rho \in \{1, \dots, r\}$,

$$q^{\rho} \in \mathcal{Z}(\mathcal{E}(q^1, \dots, q^{\rho-1})), \text{ then } F(\mathcal{E}) = F(\mathcal{E}(q^1, \dots, q^r)).$$

Lemma 2. $\mathrm{proj}_{\mathrm{span}(\mathcal{E})}(\mathcal{Z}(\mathcal{E})) = 0$ *if and only if* span$\mathcal{E}$ = pos$\mathcal{E}$.

Proof. If span$\mathcal{E}$ = pos$\mathcal{E}$, then for all $p \in \mathrm{span}\mathcal{E} \setminus \{0\}$, $\inf p \cdot X < \sup p \cdot (Y + \{\omega\})$. Thus, $\mathcal{Z}(\mathcal{E})$ is orthogonal to span$\mathcal{E}$, i.e. $\mathrm{proj}_{\mathrm{span}\mathcal{E}}(\mathcal{Z}\mathcal{E}) = 0$. Suppose span$\mathcal{E} \neq$ pos$\mathcal{E}$. Thus, $L(\mathcal{E}) \subset \mathrm{pos}\mathcal{E} \subset \mathrm{span}\mathcal{E}$ with strict inclusions. Let $\sigma \in (\mathrm{pos}\mathcal{E} \setminus \{0\}) \cap (L(\mathcal{E}))^{\perp}$. Let $z \in \mathrm{pos}\mathcal{E}$, $z_1 \in L(\mathcal{E})$ and z_2 in $(\mathrm{pos}\mathcal{E} \setminus \{0\}) \cap (L(\mathcal{E}))^{\perp}$ such that $z = z_1 + z_2$. Note that $0 \le \sigma \cdot z_2$ since otherwise there exists $\lambda \in]0, 1[$ and $\zeta = \lambda\sigma + (1 - \lambda)z_2$ such that $\sigma \cdot \zeta = 0$. Thus, $\zeta \in L(\mathcal{E}) \cap (L(\mathcal{E}))^{\perp}$ and $\zeta \neq 0$, a contradiction. Thus, $0 \le \sigma \cdot \mathrm{pos}\mathcal{E}$. Thus, $\sup \sigma \cdot (Y + \{\omega\}) \le \inf \sigma \cdot X$, and this implies that $\sigma \in \mathcal{Z}(\mathcal{E})$. Since $\sigma \neq 0$ and $\sigma \in \mathrm{span}\mathcal{E}$, thus $\mathrm{proj}_{\mathrm{span}\mathcal{E}}(\mathcal{Z}\mathcal{E}) \neq 0$. □

Given an economy $\mathcal{E}$ and a hierarchic price $\mathcal{P}$, let $s^{\mathcal{E}}(\mathcal{P}) = \min\{\sigma \in \{1, \dots, k\} | p^{\sigma} \notin \mathcal{Z}(\mathcal{E}(p^1, \dots, p^{\sigma-1}))\}$. Note that if span$\mathcal{E}$ = pos$\mathcal{E}$, and if $\mathcal{P} \subset \mathrm{span}\mathcal{E}$, then by the lemma, $s^{\mathcal{E}}(\mathcal{P}) = 1$, provided $p^1 \neq 0$.

5. Assumptions

Assumption 1. For every $i \in I$, X_i is closed, convex, $P_i : \prod_{i\in I} X_i \times \prod_{j\in J} Y_j \to 2^{X_i}$ is irreflexive, convex valued and has an open graph in $\prod_{i\in I} X_i \times \prod_{j\in J} Y_j \times X_i$.

Assumption 2. For every $j \in J$, Y_j is closed and convex.

Assumption 3. The feasible set $F(\mathcal{E})$ is bounded.

Assumption 4. For every $i \in I$, $0 \in X_i - \{\omega_i\} - \sum_{j\in J} \theta_{ij} Y_j$.

Assumption 5. For every $j \in J$ for every $y, \upsilon \in Y_j$, for every $\eta > 0$, there exists $\varepsilon > 0$ such that for all $y' \in Y_j \cap (y + \varepsilon\mathbb{B})$ there exists $\lambda \geq 1 - \eta$, $y' + \lambda(\upsilon - y) \in Y_j$.

The only non-standard assumption is Assumption 5. At first sight, it may seem surprising that unlike the standard case we use a slightly stronger assumption on the production sets than on the consumption sets. Producers can be seen as agents with a "budget" set equal to their production set for any price and with price-dependent preferences. Working with standard prices, the supply correspondence is upper semi-continuous. The lexicographic order has of course very bad continuity properties. For this reason the introduction of a production sector into a model with hierarchic prices generates additional problems. Assumption 5 ensures that the supply correspondence is upper semi-continuous with respect to a price sequence converging to a hierarchic price. A convex set satisfies Assumption 5, for example, if it is a polyhedron or if it is strictly convex. In R^2 any convex set satisfies this condition. A closed, convex set not satisfying this condition is:

$$Y = \mathrm{co}\{(1, 0, 0), (\sqrt{1-\lambda^2}, \lambda, 1) | \lambda \in [0, 1]\}.$$

This set is the convex hull of a quarter of a disc and the point $(1, 0, 0)$. To see that this set does not satisfy Assumption 5, set $y = (1, 0, 1)$, $\upsilon = (1, 0, 0)$ and take the sequence $(y^n)_{n\in N} = (\sqrt{(1 - (1/n^2))}, 1/n, 1)_{n\in N}$. For any $\lambda \neq 0$, $n \in N$, $y^n + \lambda(\upsilon - y) \notin Y$.

6. Existence

Theorem 1. *For every economy $\mathcal{E}$ satisfying* Assumptions 1–5 *there exists a hierarchic equilibrium.*

More precisely, for every $(\delta_i) \in R^I_{++}$ there exist a hierarchic equilibrium $(x, y, \mathcal{P}, w)$, $r_j \in \{1, \ldots, k\}$ and $q_j \in [0, +\infty]$ for all $j \in J \cup \{0\}$ such that for all $j \in J$ $r_j \geq 2$ and for all $i \in I$

$$w_i = \mathcal{P}\left(\omega_i + \sum_{j\in J} \theta_{ij} y_j\right) + e_{r_0} q_0 \delta_i + \sum_{j\in J} \theta_{ij} e_{r_j} q_j.$$

In the case of an exchange economy the slack in the consumers revenue can be interpreted as paper money (cf. Kajii, 1996), i.e. one may think of each consumer holding initially δ_i units of paper money and the value of money is q_0 in sub-market r_0, 0 in the better sub-markets and $+\infty$ in the sub-markets r with $r > r_0$. In the case of an economy with production one could think of an additional currency per firm distributed among its shareholders according to their share θ_{ij}. It may be interesting to note that for every firm j with a polyhedral production set we have $q_j = 0$ (see Footnote 5 in Step 7 of the Proof).

The proof goes as follows: first, using Lemma 2, we restrict the economy to an economy satisfying some global interiority condition proceeding as in Mertens (1989); then we compactify the economy; then we construct a sequence of perturbed equilibria adapting Bergstrom (1976), and Gale and Mas-Colell's (1975) approach; then we construct from this sequence using standard prices, the hierarchic price and finally in the different claims we check that the limit of our sequence satisfies the required conditions.

Proof. We start by replacing for every $i \in I$ the preference relation P_i by

$$\hat{P}_i : \prod_{i\in I} X_i \times \prod_{j\in J} Y_j \to X_i,$$

defined by $\hat{P}_i(x, y) = P_i(x, y) \cup \{\lambda x_i + (1-\lambda)\xi_i | \xi_i \in \mathrm{int}_{X_i} P_i(x, y), \lambda \in]0, 1[\}$. The new preference relation satisfies Assumption 1 and for all $(x, y) \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j$, if $\hat{P}_i(x, y) \neq \emptyset$ then $x_i \in \overline{\hat{P}_i(x, y)}$ (Gale and Mas-Colell, 1979).

Note that by Assumptions 1, 2 and 4, $F(\mathcal{E}) \neq \emptyset$. □

Step 1 (Global interiority and compactification). Let $\{p^1, \dots, p^{s-1}\} \subset R^L$ be maximal family of linearly independent vectors such that for all $r \in \{1, \dots, s-1\}$, $p^r \in \mathcal{Z}(\mathcal{E}(p^1, \dots, p^{r-1}))$. This maximal family is by Lemma 2 empty if a global interiority condition holds, i.e. if $\mathrm{pos}\mathcal{E} = R^L$. This gives us a new economy:

$$\mathcal{E}(p^1, \dots, p^{s-1}) = ((X_i(p^1, \dots, p^{s-1}), \hat{P}_i, \omega_i)_{i\in I}, (Y_j(p^1, \dots, p^{s-1}))_{j\in J}, (\theta_{ij})_{(ij)\in I\times J}).$$

Note that this economy satisfies a global interiority condition, i.e.

$$L\mathcal{E}(p^1, \dots, p^{s-1}) = \mathrm{pos}\mathcal{E}(p^1, \dots, p^{s-1}) = \mathrm{span}\mathcal{E}(p^1, \dots, p^{s-1}).$$

For $Z \in \left\{\prod_{i\in I} X_i, X_1, \dots, X_I, \prod_{j\in J} Y_j, Y_1, \dots, Y_J\right\}$, let $\hat{Z}$ be the projection of $F(\mathcal{E})$ on Z. Similarly, for such a set $Z(p^1, \dots, p^{s-1})$ in the economy $\mathcal{E}(p^1, \dots, p^{s-1})$, let $\hat{Z}(p^1, \dots, p^{s-1})$ be the projection of $F(\mathcal{E}(p^1, \dots, p^{s-1}))$ on $Z(p^1, \dots, p^{s-1})$. For all $Z \in \{X_1, \dots, X_I, Y_1, \dots, Y_J\}, \hat{Z} = \hat{Z}(p^1, \dots, p^{s-1})$ and $F(\mathcal{E}) = F(\mathcal{E}(p^1, \dots, p^{s-1}))$.

Now we will compactify the economy $\mathcal{E}(p^1, \dots, p^{s-1})$. Let $C \subset R^L$ be a convex compact set such that for all $Z \in \{X_1, \dots, X_I, Y_1, \dots, Y_J\}$, $\hat{Z} \subset \mathrm{int}C$. Such a set exists by Assumption 3. Note $\tilde{Z} = Z(p^1, \dots, p^{s-1}) \cap C$.

Step 2 (Perturbed fixed points). For all $p \in \mathbb{B}$, and for every $j \in J$ let,

$$\tilde{S}_j(p) = \mathrm{argmax}\{p \cdot y_j | y_j \in \tilde{Y}_j\}.$$

$$\varphi_j(x, y, p) = \{y' \in \tilde{Y}_j | p \cdot y' > p \cdot y_j\}.$$

Similarly, as in Bergstrom (1976) we set for all $p \in \mathbb{B}$, and for every $i \in I$,

$$B_i(p)=\left\{x_i \in X_i(p^1,\dots,p^{s-1}) | p\cdot x_i \le p\cdot\left(\omega_i+\sum_{j\in J}\theta_{ij}\tilde{S}_j(p)\right)+\delta_i(1-\|p\|)\right\}.$$

Finally, following Gale and Mas-Colell (1975, 1979), we define the correspondence φ_i from $\prod_{i\in I}\tilde{X}_i \times \prod_{j\in J}\tilde{Y}_j \times \mathbb{B}$ to X_i as follows

$$\varphi_i(x,y,p)=\begin{cases} B_i(p)\cap C, & \text{if } x_i \notin B_i(p),\\ B_i(p)\cap C\cap \hat{P}_i(x,y), & \text{if } x_i \in B_i(p).\end{cases}$$

We now consider a last agent which will revise prices maximizing the value of the excess demand. We define for all $\varepsilon \in [0, 1]$ the correspondence

$$W^\varepsilon : \prod_{i\in I}X_i \times \prod_{j\in J}Y_j \times \mathbb{B} \to \mathbb{B},$$

$$W^\varepsilon(x,y,p)=\left\{q \in (1-\varepsilon)\mathbb{B}\cap \operatorname{span}\mathcal{E}(p^1,\dots,p^{s-1}) | \right.$$

$$\left.(q-p)\cdot\left(\sum_{i\in I}x_i-\sum_{j\in J}y_j-\sum_{i\in I}\omega_i\right)>0\right\}.$$

For all $\varepsilon \in [0, 1]$ define the correspondence Ψ^ε from $\prod_{i\in I}\tilde{X}_i \times \prod_{j\in J}\tilde{Y}_j \times (1-\varepsilon)\mathbb{B}\cap \operatorname{span}\mathcal{E}(p^1,\dots,p^{s-1})$ to itself by

$$\Psi^\varepsilon(x,y,p)=\prod_{i\in I}\varphi_i(x,y,p)\times\prod_{j\in J}\varphi_j(x,y,p)\times W^\varepsilon(x,y,p).$$

For all $\varepsilon \in]0, 1]$ Ψ^ε is lower semi-continuous, convex valued and for all $i \in I\cup J$, φ_i is irreflexive and W^ε is irreflexive. So by Florenzano (1981, pp. 89–90) (see also Florenzano (1994)), for every $\varepsilon \in]0, 1]$ there exists some $(x^\varepsilon, y^\varepsilon, p^\varepsilon)$, such that

$$x_i^\varepsilon \in B_i(p^\varepsilon)\cap C \text{ and } \hat{P}_i(x^\varepsilon,y^\varepsilon)\cap B_i(p^\varepsilon)\cap C=\emptyset \text{ for all } i\in I,$$

$$y_j^\varepsilon \in \tilde{S}_j(p^\varepsilon) \text{ for all } j\in J,$$

$$p^\varepsilon \in \operatorname{argmax}\left\{q\cdot\left(\sum_{i\in I}x_i^\varepsilon-\sum_{j\in J}y_j^\varepsilon-\omega\right) | q\in(1-\varepsilon)\mathbb{B}\cap\operatorname{span}\mathcal{E}(p^1,\dots,p^{s-1})\right\}.$$

We can take a sequence $(\varepsilon = 1/n)$ with $n \in N$ and denote a corresponding sequence of perturbed equilibria by (x^n, y^n, p^n).

Case 1. Suppose for some $n \in N$, $\|p^n\| < 1-(1/n)$ then

$$x^n, y^n, \mathcal{P}, \mathcal{P}\left(\omega_i + \sum_{j\in J}\theta_{ij} y_j^n\right) + e_s\delta_i(1 - \|p^n\|),$$

with $\mathcal{P} = \{p^1, \dots, p^{s-1}, p^n\}$ is a hierarchic equilibrium.

Proof. Since the price maximizes the aggregate excess demand and $\|p^n\| \neq 1 - (1/n)$ we have

$$\sum_{i\in I} x_i^n = \sum_{j\in J} y_j^n + \omega.$$

By standard arguments:

$$y_j^n \in \operatorname{argmax}\{p \cdot y_j | y_j \in Y_j(p^1, \dots, p^{s-1})\} = S_j(\mathcal{P}),\ \forall j \in J,$$
$$x_i^n \in B_i(p^n) \text{ and } \hat{P}_i(x^n, y^n) \cap B_i(p^n) = \emptyset,\quad \forall i \in I,$$

and to conclude note that $B_i(p^n) = B_i(\mathcal{P}, \mathcal{P}(\omega_i + \sum_{j\in J}\theta y_j^n) + e_s\delta_i(1 - \|p^n\|))$. □

Case 2. For all $n \in N$, $\|p^n\| = 1 - (1/n)$.

Let N_{s-1} be an infinite subset of N such that

$$\lim_{n\to+\infty, n\in N_{s-1}} (x^n, y^n, p^n) = (x, y, p),$$

and such that for every $i \in I$,

$$\lim_{n\to+\infty, n\in N_{s-1}} B_i(p^n) = B_i,$$

in the sense of Painlevé–Kuratowski. The set B_i is non-empty by Assumption 4, closed and convex by Assumption 1 (cf. Rockafellar and Wets, 1998; Proposition 4.15, Theorem 4.18). These sets B_i will correspond to the budget sets.

Step 3. Feasibility.

Proof. Of course $\|p\| = 1$. One easily checks that

$$p \in \operatorname{argmax}\left\{q \cdot \left(\sum_{i\in I} x_i - \sum_{j\in J} y_j - \omega\right) | q \in \mathbb{B} \cap \operatorname{span}\mathcal{E}(p^1, \dots, p^{s-1})\right\}.$$

For all $n \in N_{s-1}$,

$$\|p^n\| \left\| \left(\sum_{i\in I} x_i^n - \sum_{j\in J} y_j^n - \omega\right) \right\| = p^n \cdot \left(\sum_{i\in I} x_i^n - \sum_{j\in J} y_j^n - \omega\right) \leq I(1 - \|p^n\|).$$

Thus, $\sum_{i\in I} x_i = \sum_{j\in J} y_j + \omega$. □

Step 4. For all $i \in I$, $x_i \in B_i$ and $\hat{P}_i(x, y) \cap B_i = \emptyset$.

Proof. By the definition of the Painlevé–Kuratowski limit, $x_i \in B_i$. Suppose there exists $\xi_i \in \hat{P}_i(x, y) \cap B_i$. Thus, there exists a sequence (ξ_i^n) such that for all $n \in N_{s-1}$,

$$p^n \cdot \xi_i^n < p^n \cdot \left(\omega_i + \sum_{j \in J} \theta_{ij} \tilde{S}_j(p^n)\right) + \delta_i(1 - \|p^n\|).$$

Since $\hat{P}_i$ has an open graph for all large n, $\xi_i^n \in \hat{P}_i(x^n, y^n)$. We may assume that for all $n \in N_{s-1}$, ξ_i^n is in the relative interior of $\hat{P}_i(x^n, y^n) \cap X_i(p^1, \dots, p^{s-1})$. For every $n \in N_{s-1}$, let $\lambda^n \in]0, 1]$ small enough such that $z_i^n = \lambda^n \xi_i^n + (1 - \lambda) x_i^n \in \mathrm{int} C \cap X_i(p^1, \dots, p^{s-1})$. Thus, $z_i^n \in B_i(p^n)$ and for $n \in N_{s-1}$ large enough $z_i^n \in \hat{P}_i(x^n, y^n) \cap \tilde{X}_i$. This contradicts the maximality of x_i^n. □

Step 5 (Construction of $\mathcal{P}$). Set $p_s^n = p^n$ and $p^s = p$. For $r \in \{s, \dots, L\}$ let

$$\mathcal{H}^r = \{x \in R^L | p^r \cdot x = 0\}.$$

For $r \in \{s + 1, \dots, L\}$, let

$$p_r^n = \mathrm{proj}_{\mathcal{H}^{r-1}}(p_{r-1}^n).$$

If for all big enough $n \in N_{r-1}$ $p_r^n \neq 0$, then let

$$p^r = \lim_{n \to +\infty, n \in N_r} \frac{p_r^n}{\|p_r^n\|},$$

for some infinite set $N_r \subset N_{r-1}$. If for some infinite set $N_r \subset N_{r-1}$, $p_r^n = 0$, then $p^r = \cdots = p^L = 0$. Let $k = \min\{r \in \{s, \dots, L\} | p^{r+1} = \cdots = p^L = 0\}$. So k is at most equal to L. Note that for all $r \in \{s, \dots, k\}$, $\|p_{r+1}^n\| = \|p_r^n\| o(\|p_r^n\|)$.[4]

We can thus decompose the sequence p^n in the following way:

$$p^n = \sum_{r=s}^{k} (\|p_r^n\| - \|p_{r+1}^n\|) p^r = \sum_{r=s}^{k} \varepsilon_r^n p^r,$$

with $\varepsilon_{r+1}^n = \varepsilon_r^n o(\varepsilon_r^n)$ for $r \in \{s, \dots, k - 1\}$, and ε_s^n converges to 1.

Step 6. For all $j \in J$, $y_j \in S_j(\mathcal{P})$.

Proof. Suppose for some $j \in J$, there exists $\bar{y}_j \in Y_j(p^1, \dots, p^{s-1})$ such that $\mathcal{P}\bar{y}_j > \mathcal{P}y_j$. Let $t_j = \bar{y}_j - y_j$, then for every $n \in N_k$ big enough, $p^n \cdot t_j > 0$. By Assumption 5 and the convexity of Y_j, for every $1 \geq \eta > 0$, there exists $\bar{n}$, such that for all $n \in N_k$, $n \geq \bar{n}$, and all $\lambda \in]0, \eta]$ $y_j^n + \lambda t_j \in Y_j(p^1, \dots, p^{s-1})$. Clearly for all large $n \in N_k$, $p^n \cdot (y_j^n + \lambda t_j) > p^n \cdot y_j^n$ and for $\lambda \in]0, \eta]$ small enough $y_j^n + \lambda t_j \in \tilde{Y}_j$ leading to a contradiction. □

[4] Throughout the paper we denote by $o : R \to R$ a function which is continuous at 0 with $o(0) = 0$.

Step 7 (Construction of the hierarchic revenue). For all $r \in \{s, \dots, k\}$ let

$$q_0^r = \lim_{n\to+\infty, n\in N_{k+1}} \frac{1-\|p^n\|}{\varepsilon_r^n},$$

and for all $j \in J$ let [5]

$$q_j^r = \lim_{n\to+\infty, n\in N_{k+1}} \frac{p^n \cdot (y_j^n - y_j)}{\varepsilon_r^n},$$

where N_{k+1} is an infinite subset of N_k such that the above limits exist in $\bar{R}$. For all $j \in J \cup \{0\}$ let $r_j = \max\{r \in \{1, \dots, k\} | \forall \rho < r, q_j^\rho = 0\}$, set $q_j = q_j^{r_j}$. Note that for all $j \in J$, $r_j \geq s+1$. For all $i \in I$ set

$$w_i = \mathcal{P}\left(\omega_i + \sum_{j\in J}\theta_{ij} y_j\right) + e_{r_0} q_0 \delta_i + \sum_{j\in J}\theta_{ij} e_{r_j} q_j,$$

and let $d_i = e_{r_0} q_0 \delta_i + \sum_{j\in J}\theta_{ij} e_{r_j} q_j$.

For all $i \in I$, let $\underline{r}_i = \max\{r \in \{0, \dots, k\} | \operatorname{argmin}\{p^0, \dots, p^r\} X_i \neq \emptyset\}$ with $p^0 = 0$. Let $\zeta_i \in X_i$ such that $\zeta_i \in \operatorname{argmin}\{p^0, \dots, p^{\underline{r}_i}\} X_i$ and

$$p^{\underline{r}_i+1} \cdot \left(\zeta_i - \omega_i - \sum_{j\in J}\theta_{ij} y_j\right) < 0.$$

Step 8. For every $i \in I$, we have $B_i(\mathcal{P}, w_i) \subset B_i$.

Proof. Let $\bar{r}_i$ be the smallest r such that $d_i^r > 0$. Let $\xi \in \{z \in X_i | \mathcal{P}z \leq w_i\}$. Let

$$\varrho_i^{\bar{r}_i} = \begin{cases} d_i^{\bar{r}_i}, & \text{if } d_i^{\bar{r}_i} \text{ is finite,} \\ p^{\bar{r}_i} \cdot \left(\xi - \omega_i - \sum_{j\in J}\theta_{ij} y_j\right) + 1, & \text{if } d_i^{\bar{r}_i} = +\infty. \end{cases}$$

Thus, there exists n_1 such that for all $n \in N_{k+1}, n \geq n_1$,

$$\sum_{r=s}^{k} \varepsilon_r^n p^r \cdot \left(\xi - \omega_i - \sum_{j\in J}\theta_{ij} y_j\right) \leq \varepsilon_{\bar{r}_i}^n \varrho_i^{\bar{r}_i} + \varepsilon_{\bar{r}_i}^n o(\varepsilon_{\bar{r}_i}^n).$$

There exists $M > 0$ and n_2 such that for all $n \in N_{k+1}, n \geq n_2$,

$$\sum_{r=s}^{k} \varepsilon_r^n p^r \cdot \left(\zeta - \omega_i - \sum_{j\in J}\theta_{ij} y_j\right) < \varepsilon_{\bar{r}_i}^n (\varrho_i^{\bar{r}_i} - M).$$

[5] Note that if Y_j is a polyhedron, then one may prove by induction that for all large n and for all $r \in \{s, \dots, k\}$, $\operatorname{argmax} p^n Y_j(p^1, \dots, p^{s-1}) \subset \operatorname{argmax} \sum_{\rho=s}^{r} \varepsilon_\rho^n p^\rho Y_j(p^1, \dots, p^{s-1}) \subset \operatorname{argmax}\{p^1, \dots, p^r\} Y_j$. Thus, for all large n, $\mathcal{P} y_j^n = \mathcal{P} y_j$, thus $p^n \cdot (y_j^n - y_j) = 0$.

For all $\lambda \in [0, 1[$ let $z_\lambda = \lambda\xi + (1-\lambda)\zeta$. Then for all $n \in N_{k+1}$, $n \geq \max\{n_1, n_2\}$,

$$\sum_{r=s}^{k} \varepsilon_r^n p^r \cdot \left(z_\lambda - \omega_i - \sum_{j \in J} \theta_{ij} y_j \right) < \varepsilon_{\bar{r}_i}^n (\varrho_i^{\bar{r}_i} + \lambda o(\varepsilon_{\bar{r}_i}^n) - (1-\lambda)M).$$

For all $\lambda \in [0, 1[$, there exists n_λ such that for all $n \in N_{k+1}$, $n \geq n_\lambda$,

$$\varepsilon_{\bar{r}_i}^n (\varrho_i^{\bar{r}_i} + \lambda o(\varepsilon_{\bar{r}_i}^n) - (1-\lambda)M) < \delta_i (1 - \|p^n\|) + \sum_{j \in J} \theta_{ij} p^n (y_j^n - y_j).$$

Thus, for all $n \in N_{k+1}$, $n \geq n_\lambda$,

$$\sum_{r=s}^{k} \varepsilon_r^n p^r \cdot \left(z_\lambda - \omega_i - \sum_{j \in J} \theta_{ij} y_j \right) \leq \delta_i (1 - \|p^n\|) + \sum_{j \in J} \theta_{ij} p^n (y_j^n - y_j).$$

Thus, $z_\lambda \in B_i$. Since B_i is closed $\xi \in B_i$, thus $\overline{\{z \in X_i | \mathcal{P}z \leq w_i\}} \subset B_i$ and again by the closure of B_i and by Lemma 2, $B_i(\mathcal{P}, w_i) = \overline{\{z \in X_i | \mathcal{P}z \leq w_i\}} \subset B_i$. □

Step 9. For every $i \in I$, we have $B_i \subset B_i(\mathcal{P}, w_i)$.

Proof. Let $z \in B_i$. Then there exists $z^n \subset X_i(p^1, \ldots, p^{s-1})$ converging to z such that for every $n \in N_{k+1}$,

$$p^n \cdot \left(z^n - \omega_i - \sum_{j \in J} \theta_{ij} y_j^n \right) \leq \delta_i (1 - \|p^n\|).$$

For every $\lambda \in [0, 1[$, let $z_\lambda = \lambda z + (1-\lambda)\zeta$. Note that ζ may have been chosen such that $\mathcal{P}z_\lambda \leq \mathcal{P}z^n$ for all $n \in N_{k+1}$ large enough. Thus, there exists $n_\lambda \in N$, such that for all $n \in N_{k+1}$ with $n \geq n_\lambda$,

$$p^n \cdot \left(z_\lambda - \omega_i - \sum_{j \in J} \theta_{ij} y_j \right) \leq \delta_i (1 - \|p^n\|) + \sum_{j \in J} \theta_{ij} p^n \cdot (y_j^n - y_j).$$

Let $r = \max\left\{\rho \in \{1, \ldots, k\} | \{p^1, \ldots, p^{\rho-1}\} \left(z_\lambda - \omega_i - \sum_{j \in J} \theta_{ij} y_j \right) = 0 \right\}$. Thus,

$$\sum_{\rho=r}^{k} \frac{\varepsilon_\rho^n}{\varepsilon_r^n} p^\rho \left(z_\lambda - \omega_i - \sum_{j \in J} \theta_{ij} y_j \right) < \frac{\delta_i (1 - \|p^n)\| + \sum_{j \in J} \theta_{ij} p^n \cdot (y_j^n - y_j)}{\varepsilon_r^n}.$$

Then, going to the limit one obtains

$$p^r \left(z_\lambda - \omega_i - \sum_{j \in J} \theta_{ij} y_j \right) \leq \lim_{n \to +\infty, n \in N_{k+1}} \frac{\delta_i (1 - \|p^n\|) + \sum_{j \in J} \theta_{ij} p^n \cdot (y_j^n - y_j)}{\varepsilon_r^n} = d_i^r.$$

Either $p^r \cdot \left(z_\lambda - \omega_i - \sum_{j\in J}\theta_{ij}y_j\right) < 0$ and then $z_\lambda \in B_i(\mathcal{P}, w_i)$ or $p^r \cdot \left(z_\lambda - \omega_i - \sum_{j\in J}\theta_{ij}y_j\right) > 0$. In the latter case either $0 < p^r \left(z_\lambda - \omega_i - \sum_{j\in J}\theta_{ij}y_j\right) \leq d_i^r$ or for some $\rho < r$, $d_i^\rho > 0$. Anyhow $r_i(\mathcal{P}, w_i) \leq r$ and $z_\lambda \in B_i(\mathcal{P}, w_i)$. Since for all $\lambda \in [0, 1[$, $z_\lambda \in B_i(\mathcal{P}, w_i)$, $z \in B_i(\mathcal{P}, w_i)$ by the closure of $B_i(\mathcal{P}, w_i)$. □

7. Walras equilibrium

We give now conditions ensuring that a hierarchic equilibrium is a Walras equilibrium of some reduced economy.

Assumption 6. $\mathrm{pos}(X_1 - \{\omega_1\} - \sum_{j\in J}\theta_{1j}Y_j) = \cdots = \mathrm{pos}(X_I - \{\omega_I\} - \sum_{j\in J}\theta_{Ij}Y_j)$.

Assumption 7. For every $i \in I$, for every $(x, y) \in F(\mathcal{E})$, $x_i \in \overline{P_i(x, y) \cap (L(\mathcal{E}) + \{x_i\})}$.

Assumption 6 ensures that all agents have an income of the same order. It is implied by a strong survival assumption. Assumption 7 states that all agents are locally non-satiated in the lineality of $\mathcal{E}$, i.e. the largest subspace where a global interiority assumption holds. It is implied by a non-satiation assumption together with a global interiority assumption. One could weaken these conditions by imposing irreducibility (Hammond, 1993; Florig, 2000) on the lineality of $\mathcal{E}$ rather than on its linear span.

Proposition 2. *Suppose the economy $\mathcal{E}$ satisfies* Assumptions 6 *and* 7. *Then, for every hierarchic equilibrium $(x, y, \mathcal{P}, w)$, the collection $(x, y, p^{s^{\mathcal{E}}(\mathcal{P})})$ is a Walras equilibrium of the economy $\mathcal{E}(p^1, \ldots, p^{s^{\mathcal{E}}(\mathcal{P})-1})$.*

Proof. Note first that there exists $r \leq k$ such that $p^r \notin \mathcal{Z}(\mathcal{E}(p^1, \ldots, p^{r-1}))$. Indeed, note that, by Assumption 7, $L(\mathcal{E})$ is of dimension at least one. If for all $r \in \{1, \ldots, k\}$, $p^r \in \mathcal{Z}(\mathcal{E}(p^1, \ldots, p^{r-1}))$, then $\{p^1, \ldots, p^k\}$ are all orthogonal to $L(\mathcal{E})$ and for all $i \in I$,

$$\emptyset \neq (L(\mathcal{E}) + \{x_i\}) \cap X_i \subset B_i(\mathcal{P}, w_i),$$

contradicting the maximality of x_i. Set $s = s^{\mathcal{E}}(\mathcal{P})$ in order to simplify notations. Note that for all $i \in I$,

$$\sup\{p^1, \ldots, p^s\}\left(\{\omega_i\} + \sum_{j\in J}\theta_{ij}Y_j\right) > \inf\{p^1, \ldots, p^s\}X_i.$$

Indeed, this is satisfied at least for one $i \in I$ and then, by Assumption 6, it is satisfied for all $i \in I$. Then, for all $i \in I$,

$$B_i(\{p^1, \ldots, p^s\}, \{p^1, \ldots, p^s\}\omega_i + \sum_{j\in J}\theta_{ij}\pi_i(p^1, \ldots, p^s)) \subset B_i(\mathcal{P}, w_i).$$

Furthermore, $L(\mathcal{E}) = L(\mathcal{E}(p^1, \dots, p^{s-1}))$. Thus, every $i \in I$ is locally non-satiated in $X_i(p^1, \dots, p^{s-1})$ at (x, y). Thus, for all $i \in I$,

$$\{p^1, \dots, p^s\}\omega_i + \sum_{j \in J} \theta_{ij}\pi_i(p^1, \dots, p^s) \leq \{p^1, \dots, p^s\}x_i = (w_i^1, \dots, w_i^s),$$

and by feasibility the inequality holds as an equality.

Hence (x, y, p^s) is a Walras equilibrium of $\mathcal{E}(p^1, \dots, p^{s-1})$. □

As $F(\mathcal{E}) = F(\mathcal{E}(p^1, \dots, p^{s^{\mathcal{E}}(\mathcal{P})-1}))$, (x, y) is Pareto optimal and in the core of $\mathcal{E}$.

Corollary 1. *Let $\mathcal{E}$ satisfy* Assumptions 6 and 7 *and suppose* $\text{pos}\mathcal{E} = \text{span}\mathcal{E}$. *Then, for any hierarchic equilibrium $(x, y, \mathcal{P}, w)$, the collection (x, y, p^r) is a Walras equilibrium with* $r = \text{argmin}\{\rho \in \{1, \dots, k\} | p^\rho \notin (\text{span}\mathcal{E})^\perp\}$.

8. Welfare analysis

We will first prove weak Pareto optimality of hierarchic equilibria. Then, we will show that every weak Pareto optimum (and hence any Pareto optimum) can be decentralized by a hierarchic price and revenue without the standard border condition (cf. Debreu, 1959). Similar results have been given by Marakulin (1990) for exchange economies within terms of non-standard analysis. Later, we show that hierarchic equilibria where all agents minimize expenditure (cf. Mas-Colell, 1992) are Pareto optimal and we prove their existence under additional assumptions.

Definition 5.

1. A collection $(x, y) \in F(\mathcal{E})$ is *weakly Pareto optimal*, if there does not exist $(x', y') \in F(\mathcal{E})$ with $x \neq x'$ and $x'_i \in P_i(x, y)$ for all $i \in I$ such that $x_i \neq x'_i$.
2. A collection $(x, y) \in F(\mathcal{E})$ is *Pareto optimal*, if there does not exist $(x', y') \in F(\mathcal{E})$ such that for some $i \in I$, $x'_i \in P_i(x, y)$ and for all $i \in I$ $x_i \notin P_i(x', y')$.

Note that our definition of weak Pareto optimality differs from the usual one according to which an allocation is weakly Pareto optimal, if no allocation preferred by every consumer exist. This definition would have the defect, that if one consumer is indifferent between any allocation, i.e. he is not interested in anything, then any feasible allocation is weakly Pareto optimal. The definition of Pareto optimality is a straightforward generalization to the case of intransitive and incomplete preferences. It differs however from the definitions of Pareto optimality usually encountered in the literature on such preferences. These are in fact refinements of weak Pareto optimality which do not reduce to the standard definition of Pareto optimality when preferences are assumed to be complete and transitive (see Florenzano (1981) for a survey).

Proposition 3. *Every hierarchic equilibrium $(x, y, \mathcal{P}, w)$ is weakly Pareto optimal.*

Proof. Suppose (x, y) is not weakly Pareto optimal. Then, there exists $(x', y') \in F(\mathcal{E})$ with $x' \neq x$ and $x'_i \in P_i(x, y)$ for all $i \in I$ such that $x_i \neq x'_i$. Therefore, for all $i \in I$, such that $x_i \neq x'_i$, $\mathcal{P}x'_i > \mathcal{P}x_i$ and thus

$$\mathcal{P}\sum_{i\in I} x'_i > \mathcal{P}\sum_{i\in I} x_i = \mathcal{P}\left(\sum_{j\in J} y_j + \omega\right) \geq \mathcal{P}\left(\sum_{j\in J} y'_j + \omega\right).$$

Therefore, (x', y') is not feasible. □

Under a global interiority and local non-satiation condition there are no Pareto dominating allocations where the richest consumers (i.e. those with non minimum-wealth with respect to p^1) are better off. However, Pareto dominating allocations may exist where consumers with minimum-wealth with respect to p^1 receive strictly preferred allocations.

Example 5. In the standard framework the second welfare theorem requires a border condition which does not hold in this example. Let $X_1 = X_2 = R^2_+$, $u_1(x) = x^1\sqrt{x^1}$, $u_2(x) = \sqrt{(x^1)} + x^2$, the total endowment is (1, 1) and $x_1 = (1, 0)$, $x_2 = (0, 1)$. Clearly, (x_1, x_2) is a Pareto optimum and all standard conditions are satisfied apart the fact that the allocation is not interior. It cannot be decentralized by a standard price. Indeed, both components of a decentralizing price would need to be strictly positive, but then consumer 2 would prefer to sell some of good 2 in order to buy some of good 1. The hierarchic price $\mathcal{P} = \{(1, 0), (0, 1)\}$ decentralizes the Pareto optimum.

Proposition 4. *Suppose for all* $i \in I$, X_i *is convex,* P_i *is irreflexive, convex valued and it has open values in* X_i*. For all* $j \in J$, Y_j *is convex. Let* $(x, y) \in F(\mathcal{E})$ *be (weakly) Pareto optimal, then there exists* $\mathcal{P}$, (ω'_i), (θ'_{ij}) with $\sum_{i\in I}\omega'_i = \omega$, *such that the collection* $\left(x, y, \mathcal{P}, \left(\mathcal{P}\omega'_i + \sum_{j\in J}\theta'_{ij}\pi_j(\mathcal{P})\right)_{i\in I}\right)$ *is a hierarchic equilibrium of the economy* $\mathcal{E}' = ((X_i, P_i, \omega'_i), (Y_j), (\theta'_{ij}))$.

Proof. Choose some $i' \in I$, then for $i \in I \setminus \{i'\}$ set $\omega'_i = x_i$ and for all $j \in J$, $\theta_{ij} = 0$. Set $\omega_{i'} = \omega - \sum_{i\in I\setminus\{i'\}} x_i$ and for all $j \in J$, $\theta_{i'j} = 1$. For a sequence of vectors $q^1, \dots, q^L$ and for $r \in \{1, \dots, L\}$, let

$$I^{r-1} = I(q^1, \dots, q^{r-1}) = \{i \in I \,|\, P_i(x, y) \cap X_i(q^1, \dots, q^{r-1}) \neq \emptyset\},$$

$$G(q^1, \dots, q^{r-1}) = \sum_{i\in I^{r-1}} P_i(x, y) \cap X_i(q^1, \dots, q^{r-1}) + \sum_{i\notin I^{r-1}} x_i - \sum_{j\in J} Y_j(q^1, \dots, q^{r-1}) - \omega.$$

Note that $G(q^1, \dots, q^{r-1})$ is convex. Let $r \in \{1, \dots, L\}$, then if $I(p^1, \dots, p^{r-1}) = \emptyset$ set $\mathcal{P} = \{p^1, \dots, p^{r-1}\}$ if $r \geq 2$ and $\mathcal{P} = 0$ otherwise. If $I(p^1, \dots, p^{r-1}) \neq \emptyset$, then 0 is not in the relative interior of $G(p^1, \dots, p^{r-1})$. We can therefore choose some vector $p^r \neq 0$ in the vector subspace spanned by $G(p^1, \dots, p^{r-1})$ which is orthogonal to the vector subspace spanned by $\{p^1, \dots, p^{r-1}\}-$ such that $0 \leq \inf p^r \cdot G(p^1, \dots, p^{r-1})$. Now for every $j \in J$, $\{p^1, \dots, p^r\}y_j = \pi_j(p^1, \dots, p^r)$ and for $i \in I$,

$$\{p^1, \dots, p^r\}x_i > \inf\{p^1, \dots, p^r\}X_i \Rightarrow i \notin I(p^1, \dots, p^r).$$

For some $k \leq L$, the process stops, $I(\mathcal{P}) = \emptyset$ and for all $i \in I$,

$$P_i(x, y) \cap B_i\left(\mathcal{P}, \mathcal{P}\omega_i' + \sum_{j \in J}\theta_{ij}'\pi_j(\mathcal{P})\right) = \emptyset.$$

□

We will now turn to the existence of Pareto optimal hierarchic equilibria. For this we define the notion of expenditure minimization (cf. Mas-Colell, 1992) for our set-up.

Definition 6. Consumer $i \in I$ minimizes expenditure at $(x, y, \mathcal{P}) \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j \times \mathcal{HP}$ if for all $(x', y') \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j$, $x_i \notin P_i(x', y')$ implies $\mathcal{P}x_i' \geq \mathcal{P}x_i$.

Proposition 5. *Every hierarchic equilibrium* $(x, y, \mathcal{P}, w)$ *satisfying expenditure minimization for all* $i \in I$ *is Pareto optimal*.

Proof. Let $(x', y') \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j$ Pareto dominating (x, y). So, for all $i \in I$, by expenditure minimization, $\mathcal{P}x_i' \geq \mathcal{P}x_i$ and for at least one $i \in I$, $\mathcal{P}x_i' > w_i \geq \mathcal{P}x_i$. Hence, $\mathcal{P}\sum_{i\in I} x_i' > \mathcal{P}\sum_{i\in I} x_i = \mathcal{P}\left(\sum_{j\in J} y_j + \omega\right) \geq \mathcal{P}\left(\sum_{j\in J} y_j' + \omega\right)$. Thus, $(x', y') \notin F(\mathcal{E})$. □

We show under additional assumptions the existence of a hierarchic equilibrium with expenditure minimization for all $i \in I$. For all $i \in I$, let $K_i = \mathrm{proj}_{X_i}\left\{(x, y) \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j | P_i(x, y) = \emptyset\right\}$ be the projection of the satiation points on X_i.

Assumption 8. For all $i \in I$,

1. K_i is convex and closed,
2. If $P_i(x, y) = \emptyset$, then for all $(x', y') \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j$ such that $P_i(x', y') \neq \emptyset$, we have $x_i \in P_i(x', y')$.

Assumption 9. For all $i \in I$ and all $(x, y), (x', y') \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j$ such that $P_i(x', y') \neq \emptyset$, if $x_i \notin P_i(x', y')$, then $x_i' \in \overline{P_i(x, y)}$.

Assumption 10. For every $i \in I$ for every $(x, y), (x', y') \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j$ such that $x_i \notin P_i(x', y')$, there exists $\bar{\varepsilon} > 0$ such that for every $\varepsilon \in]0, \bar{\varepsilon}]$ and every

$$(\xi, \upsilon) \in \prod_{i\in I}(X_i \cap (x_i + \varepsilon\mathbb{B})) \times \prod_{j\in J}(Y_j \cap (y_j + \varepsilon\mathbb{B})),$$

there exists $\lambda \in]0, 1]$ such that either $\bar{\xi}_i = \xi_i + \lambda(x'_i - x_i) \in \overline{P_i(\xi, \upsilon)}$, or $P_i(\bar{\xi}, \upsilon) = \emptyset$ with $\bar{\xi}_{i'} = \xi_{i'}$ for all $i' \neq i$.

Assumption 8 states that satiation points are always in the preferred set of points of non-satiation. Assumption 9 is fulfilled if preferences are complete and satisfying local non-satiation at the points of non-satiation. Assumption 10 requires Assumption 5 to hold also for all the consumption sets. Moreover, some kind of local homogenity is needed in the affine subspaces where the preferred set is not strictly convex. A preference relation not satisfying X is generated by the following:

Let $P : R^2_+ \to R^2_+$ with an open graph, irreflexive, transitive, satisfying Assumption 9 and such that for all $t \in R_+$, $P(t, 0) = \{y \in R^2_+ | y_1 + y_2\sqrt{t} > t\}$. Then, $(1, 0) \notin P(0, 1)$, but for all $\varepsilon > 0$ and all $\lambda \in]0, 1]$, $(1 + \varepsilon, 0) + \lambda(-1, 1) \notin \overline{P(1 + \varepsilon, 0)}$.

Theorem 2. *For every economy $\mathcal{E}$ satisfying* Assumptions 1–5 *and* 8–10 *there exists a hierarchic equilibrium such that all consumers $i \in I$ minimize expenditure.*

More precisely, for every $(\delta_i) \in R^I_{++}$ there exist such a hierarchic equilibrium $(x, y, \mathcal{P}, w)$, $r_j \in \{1, \ldots, k\}$ and *$q_j \in [0, +\infty]$ for all $j \in J \cup \{0\}$ such that for all $j \in J$ $r_j \geq 2$ and for all $i \in I$*

$$w_i = \mathcal{P}\left(\omega_i + \sum_{j\in J}\theta_{ij}y_j\right) + e_{r_0}q_0\delta_i + \sum_{j\in J}\theta_{ij}e_{r_j}q_j.$$

Proof. We start by defining price-dependent preferences correspondences

$$\tilde{P}_i : \prod_{i\in I}X_i \times \prod_{j\in J}Y_j \times \mathbb{B} \to X_i,$$

$$\tilde{P}_i(x, y, p) = \begin{cases} P_i(x, y), & \text{if } x_i \notin K_i, \\ K_i \cap \{x'_i \in X_i | p \cdot x'_i < p \cdot x_i\}, & \text{if } x_i \in K_i. \end{cases}$$

□

Claim 1. *For all $i \in I$ $\tilde{P}_i$ is irreflexive, convex valued and lower semi-continuous.*

Proof. For all $(x, y, p) \in \prod_{i\in I}X_i \times \prod_{j\in J}Y_j \times \mathbb{B}$ such that $x_i \notin K_i$, $\tilde{P}_i$ coincides locally with P_i by Assumption 8(1). So it is irreflexive, convex valued and lower semi-continuous there.

Let $(x, y, p) \in \prod_{i\in I}X_i \times \prod_{j\in J}Y_j \times \mathbb{B}$ such that $x_i \in K_i$. Then, $\tilde{P}_i$ is irreflexive at (x, y, p) since $x_i \notin \{x'_i \in X_i | p \cdot x'_i < p \cdot x_i\}$ and it is convex valued by Assumption 8(1).

Let U be an open subset of X_i such that $U \cap \tilde{P}_i(x, y, p) \neq \emptyset$. We may assume without loss of generality that for all $x''_i \in U$, $p \cdot x''_i < p \cdot x_i - 2\varepsilon$ for some fixed $\varepsilon > 0$. Let V be open neighborhood of (x, y, p) such that for all $(x', y', p') \in V$, $p' \cdot x'_i > p' \cdot x_i - \varepsilon$ and for all $x''_i \in U$, $p' \cdot x''_i < p' \cdot x_i - \varepsilon$.

For $(x', y', p') \in V$ such that $x'_i \in K_i$ we have for all $x''_i \in U$, $p' \cdot x''_i < p' \cdot x_i - \varepsilon < p' \cdot x'_i$. Thus, $U \cap \tilde{P}_i(x', y', p') \neq \emptyset$.

For $(x', y', p') \in V$ such that $x'_i \notin K_i$ we have $P_i(x', y', p') \neq \emptyset$ thus by Assumption 8(2), $K_i \subset P_i(x', y', p')$ and thus $U \cap \tilde{P}_i(x', y', p') \neq \emptyset$.

Thus, $\tilde{P}_i$ is lower semi-continuous. □

Note that the economy with the preference correspondences P_i replaced by $\tilde{P}_i$ satisfies all assumptions of Theorem 1 apart the open graph property of the preferences. Apart in Step 4 of the theorem's proof, only lower semi-continuity of the preferences is used. Furthermore, by Assumption 9, P_i coincides with its augmentation $\hat{P}_i$. So for every $n \in N$ we have the existence of a perturbed equilibrium (x^n, y^n, p^n) as in Step 2, but now relative to the preferences $\tilde{P}_i$. Since for all $(x', y', p) \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j \times \mathbb{B}$, $P_i(x', y') \subset \tilde{P}_i(x', y', p)$, (x^n, y^n, p^n) "converges", as in the proof of Theorem 1, to a hierarchic equilibrium $(x, y, \mathcal{P}, w)$ with respect to the preferences P_i.

Claim 2. *For all $n \in N$ all consumers minimize expenditure at (x^n, y^n, p^n).*

Proof. If for some $n \in N$ and $i \in I$, $x^n_i \notin K_i$ and for some $(x', y') \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j$, $x^n_i \notin P_i(x', y')$ then by Assumption 9, $x'_i \in \overline{P_i(x^n, y^n)}$. Then,

$$p^n \cdot x'_i \geq p^n \cdot \left(\omega_i + \sum_{j\in J} \theta_{ij} \tilde{S}_j(p^n)\right) + \delta_i(1 - \|p^n\|) \geq p^n \cdot x^n_i.$$

If $x^n_i \in K_i$ and for some $(x', y') \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j$, $x^n_i \notin P_i(x', y')$ then by Assumption 8(2), $P_i(x', y') = \emptyset$, i.e. $x'_i \in K_i$ and therefore $p^n \cdot x'_i \geq p^n \cdot x^n_i$. □

Claim 3. *All consumers minimize expenditure at $(x, y, \mathcal{P})$.*

Proof. Let $i \in I$ and $(x', y') \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j$ such that $x_i \notin P_i(x', y')$ and $\mathcal{P}x'_i < \mathcal{P}x_i$. Thus, for all n, $p^n \cdot (x'_i - x_i) < 0$. By Assumption 10, for all large enough n, there exists $\lambda^n \in]0, 1]$ such that either $\xi^n_i = x^n_i + \lambda^n(x'_i - x_i) \in \overline{P_i(x^n, y^n)}$ or $P_i(\xi^n, y^n) = \emptyset$ with $\xi^n_{i'} = x^n_{i'}$ for all $i' \neq i$. Since $p^n \cdot \xi^n_i < p^n \cdot x^n_i$ this would contradict expenditure minimization at n. □

Example 6. The theorem fails without Assumption 9 (cf. Mas-Colell, 1992). Consider an exchange economy with two consumers and one commodity, so the concept of hierarchic equilibrium and dividend equilibrium coincide. Let $X_1 = X_2 = R_+$, $\omega_1 = \omega_2 = 2$, $u_1(x) = x$,

$$u_2(x) = \begin{cases} 0, & \text{if } x \leq 3, \\ x - 3, & \text{if } x \geq 3. \end{cases}$$

All assumptions except Assumption 9 are satisfied.

Then, $x_1 = 2 + \alpha$, $x_2 = 2 - \alpha$, $p = 1$, $w_1 = w_2 = 2 + \alpha$ for $\alpha \in [0, 1]$ constitutes the whole set of hierarchic equilibria with $\delta_1 = \delta_2 > 0$. For $\delta_1 = 2$, $\delta_2 \in [0, 1]$, $x_1 = 4$, $x_2 = 0$, $p = 1$, $w_1 = 4$, $w_2 = 2 + \delta_2$ is a hierarchic equilibrium satisfying expenditure minimization. It corresponds to the unique Pareto optimum. So if the slack variables (δ_i) are not carefully chosen, a Pareto optimal hierarchic equilibrium may fail to exist.

9. Core equivalence

Using Aubin's (1979) fuzzy approach, Konovalov (1998) extends Debreu and Scarf's (1963) classical core equivalence result to the case of dividend equilibria within exchange economies. For this he introduced a refinement of the core — the rejective core. We will extend this equivalence result to the hierarchic equilibrium. Core equivalence for dividend equilibria and for Walras equilibria are obtained as corollaries.

Usually, the core is defined as the set of allocations which cannot be improved upon by any coalition $\mathcal{I} \subset I$. There are two standard meanings to this:

1. The coalition $\mathcal{I}$ can achieve, on its own, an allocation they all prefer weakly and some prefer it strictly.
2. The coalition $\mathcal{I}$ can achieve, on its own, an allocation they all prefer strictly.

Without a strong survival assumption and/or without a non-satiation assumption, the core according to (1) (i.e. the strong core, that is the core concept using the weak notion of blocking) is too small (it may be empty). However, the core according to (2) (i.e. the weak core, that is the core concept using the strong notion of blocking) may be too big. To illustrate this consider again Example 3 of Section 3.

Example 7. Let $X_i = R^3_+$, $u_1(x) = x^1 - x^2 - x^3$, $u_2(x) = x^1 + 2x^2 + x^3$, $u_3(x) = x^1 + x^2 + 2x^3$, $\omega_1 = (1, 1, 1)$, $\omega_2 = \omega_3 = (0, 0, 0)$. The core according to (1) is empty and the core according to (2) contains the allocation $x_1 = (1, 0, 0)$, $x_2 = (0, 0, 1)$, $x_3 = (0, 1, 0)$. However, once this allocation is realized, agents two and three could continue to trade amongst themselves leading for example to the allocation $\xi_1 = (1, 0, 0)$, $\xi_2 = (0, 1, 0)$, $\xi_3 = (0, 0, 1)$, they both prefer. Since allocation x is not weakly Pareto optimal, by Proposition 3 it cannot be decentralized by a hierarchic price.

Following Konovalov (1998), we will define the fuzzy rejective core for our framework.

Definition 7.

1. The fuzzy coalition $(\lambda_i)_{i\in I} \in [0, 1]^I \setminus \{0\}$ *f-rejects* $(x, y) \in F(\mathcal{E})$, if there exist $\mu \in R^I_+$, $\eta \in R^I_+$ and an allocation $x' \in \prod_{i\in I} X_i$ such that:
   1.1. for all $i \in I$, $\mu_i + \eta_i = \lambda_i$;
   1.2. $\sum_{i\in I}\lambda_i x'_i \in \sum_{i\in I}\mu_i \left(x_i + \sum_{j\in J}\theta_{ij}(Y_j - y_j)\right) + \sum_{i\in I}\eta_i \left(\omega_i + \sum_{j\in J}\theta_{ij}Y_j\right)$;
   1.3. for all $i \in I$ with $\lambda_i > 0$, $x'_i \in P_i(x, y)$.
2. The *fuzzy rejective core* $\mathcal{FRCE}$ of $(\mathcal{E})$ is the set of $(x, y) \in F(\mathcal{E})$ which cannot be f-rejected by a fuzzy coalition.

In order to give an interpretation to this core concept, think of an economy with private production $Y_i = \sum_{j\in J}\theta_{ij}Y_j$ and a continuum of consumers for each type i. According to this notion of rejection, the group λ (λ_i being the measure of members of type i) may justify an objection against $(x, y) \in F(\mathcal{E})$ by the argument: "even if $[0, 1]^I \setminus [0, \lambda_1] \times \cdots \times [0, \lambda_I]$ were able to realize with some of us, i.e. with μ, the allocation $x \in \prod_{i \in I} X_i$, then once the corresponding exchanges and productions are realized, we (group λ) will change the allocation on the components we control towards some $x' \in \prod_{i \in I} X_i$ we all prefer."

Of course the fuzzy rejective core is included in the in the weaker of the two standard core concepts, i.e. the core according to (2).

In order to establish equivalence between the core allocations and the hierarchic equilibrium allocations, one needs a weaker form of profit maximization.

Example 8. Consider a two consumers one good economy with $X_1 = X_2 = R_+$, initial endowments $\omega_1 = \omega_2 = 1$ and $u_1(x) = -x$, $u_2(x) = x$. Suppose there is one firm entirely owned by consumer 1 with a production set $Y = -R_+$. No Walras equilibrium exists and there exists a unique hierarchic equilibrium $x_1 = 0$, $x_2 = 2$, $y = 0$, $p = 1$, $w_1 \geq 1$, $w_2 = 2$. However, $x_1 = 0$, $x_2 = 1 + t$, $y = t - 1$ for $t \in [0, 1]$ is in the fuzzy rejective core and together with $p = 1$, $w_1 \geq 1$, $w_2 = 1 + t$ it could be called a weak hierarchic equilibrium. Clearly, there is no reason why consumer 1's firm should maximize profit; there is no incentive to do so. This example also shows that the fuzzy rejective core is not included in the set of weak Pareto optima.

Let $(x, y, \mathcal{P}, w) \in \prod_{i \in I} X_i \times \prod_{j \in J} Y_j \times \mathcal{HP} \times (\bar{R}^k)^I$, be given. For every $i \in I$, let

$$\kappa_i = \min \{r \in \{0, 1, \ldots, k\} | P_i(x, y) \cap B_i(\mathcal{P}, (w_i^1, \ldots, w_i^r, +\infty, \ldots)) = \emptyset\}.$$

For every $j \in J$, let

$$\kappa_j = \max\{\kappa_i | i \in I, \theta_{ij} > 0\}.$$

For $\kappa \in \{0, 1, \ldots, k\}$, let $\mathcal{P}(\kappa) = 0$, if $\kappa = 0$, and $\mathcal{P}(\kappa) = \{p^1, \ldots, p^\kappa\}$ otherwise.

Definition 8. A collection $(x, y, \mathcal{P}, w) \in F(\mathcal{E}) \times \mathcal{HP} \times (\bar{R}^k)^I$ is a *weak hierarchic equilibrium* of the economy $\mathcal{E}$ if:

1. for all $i \in I$, $x_i \in B_i(\mathcal{P}, w_i)$ and $P_i(x, y) \cap B_i(\mathcal{P}, w_i) = \emptyset$;
2. for all $i \in I$, $\mathcal{P}\omega_i + \sum_{j \in J} \theta_{ij} \mathcal{P} y_j \leq w_i$;
3. for all $j \in J$, $y_j \in S_j(\mathcal{P}(\kappa_j))$.

Every hierarchic equilibrium is a weak hierarchic equilibrium. When minimum-wealth situations can be excluded, i.e. for example, if the strong survival assumption holds, then this reduces to a sort of dividend equilibrium where the profit maximizing firms are those who's shareholders are not all satiated in their preferences. If furthermore non-satiation holds, then this reduces to a Walras equilibrium.

Proposition 6. *Every weak hierarchic equilibrium is in the fuzzy rejective core.*

Proof. Let $(x, y, \mathcal{P}, w)$ be a weak hierarchic equilibrium and suppose without loss of generality that for all $i \in I$, $\mathcal{P}x_i \leq w_i$. Let $\lambda \in [0, 1]^I \setminus \{0\}$, $\mu \in R_+^I$, $\eta \in R_+^I$ and an allocation $x' \in \prod_{i \in I} X_i$ such that:

1. for all $i \in I$, $\mu_i + \eta_i = \lambda_i$;
2. for all $i \in I$ with $\lambda_i > 0$, $x'_i \in P_i(x, y)$.

Let $\mathcal{I} = \{i \in I | \lambda_i > 0\}$, $\kappa = \max\{\kappa_i | i \in \mathcal{I}\}$. Then, for every $i \in \mathcal{I}$,

$$\mathcal{P}(\kappa_i)x_i \leq w_i < \mathcal{P}(\kappa_i)x'_i.$$

For all $j \in J$ such that for some $i \in \mathcal{I}, \theta_{ij} > 0$, $y'_j \in Y_j$ implies $\mathcal{P}(\kappa_j)(y'_j - y_j) \leq 0$. Then, for every $i \in \mathcal{I}$,

$$\mathcal{P}(\kappa_i)x'_i > \max\left\{\sup\left\{\mathcal{P}(\kappa_i)\left(x_i + \sum_{j \in J}\theta_{ij}(Y_j - y_j)\right)\right\},\right.$$
$$\left.\sup\left\{\mathcal{P}(\kappa_i)\left(\omega_i + \sum_{j \in J}\theta_{ij}Y_j\right)\right\}\right\}.$$

Hence, for every $i \in \mathcal{I}$,

$$\mathcal{P}(\kappa)x'_i > \max\left\{\sup\left\{\mathcal{P}(\kappa)\left(x_i + \sum_{j \in J}\theta_{ij}(Y_j - y_j)\right)\right\},\right.$$
$$\left.\sup\left\{\mathcal{P}(\kappa)\left(\omega_i + \sum_{j \in J}\theta_{ij}Y_j\right)\right\}\right\}.$$

Therefore,

$$\mathcal{P}(\kappa)\sum_{i \in I}\lambda_i x'_i > \sup\left\{\mathcal{P}(\kappa)\left(\sum_{i \in I}\mu_i\left(x_i + \sum_{j \in J}\theta_{ij}(Y_j - y_j)\right)\right.\right.$$
$$\left.\left.+\sum_{i \in I}\eta_i\left(\omega_i + \sum_{j \in J}\theta_{ij}Y_j\right)\right)\right\},$$

and finally,

$$\sum_{i \in I}\lambda_i x'_i \notin \sum_{i \in I}\mu_i\left(x_i + \sum_{j \in J}\theta_{ij}(Y_j - y_j)\right) + \sum_{i \in I}\eta_i\left(\omega_i + \sum_{j \in J}\theta_{ij}Y_j\right).$$

Thus, $(x, y) \in \mathcal{FRC}(\mathcal{E})$. □

Proposition 7. *Suppose for every $j \in J$, Y_j is convex. Suppose for every $i \in I$, X_i is convex, for every $(\bar{x}, \bar{y}) \in \mathcal{FRC}(\mathcal{E})$, $P_i(\bar{x}, \bar{y})$ is convex and open in X_i. Moreover, if $J \neq \emptyset$, suppose that $P_i(\bar{x}, \bar{y}) \neq \emptyset$ for $i \in I$ and $(\bar{x}, \bar{y}) \in \mathcal{FRC}(\mathcal{E})$ implies $\bar{x}_i \in \overline{P_i(\bar{x}, \bar{y})}$.*[6]

Then, for every $(\bar{x}, \bar{y}) \in \mathcal{FRC}(\mathcal{E})$ there exists $(\mathcal{P}, w) \in \mathcal{HP} \times (\bar{R}^k)^I$ such that $(\bar{x}, \bar{y}, \mathcal{P}, w)$ is a weak hierarchic equilibrium.

Proof. Let $(\bar{x}, \bar{y}) \in \mathcal{FRC}(\mathcal{E})$. For every $i \in I$, let

$$G_i = P_i(\bar{x}, \bar{y}) - \{\bar{x}_i\} - \sum_{j \in J} \theta_{ij}(Y_j - \bar{y}_j); \qquad H_i = P_i(\bar{x}, \bar{y}) - \{\omega_i\} - \sum_{j \in J} \theta_{ij} Y_j;$$
$$\mathcal{K} = \mathrm{co} \cup_{i \in I} (G_i \cup H_i).$$

If $\mathcal{K} = \emptyset$, then 0 is a hierarchic equilibrium price. Otherwise, we need to proceed as follows. □

Claim 4. $0 \notin \mathcal{K}$.

Proof. We proceed by contraposition. Suppose $0 \in \mathcal{K}$. Then, there exists $\bar{N} \in N$ and vectors (x_{i_n}), (ζ_{i_n}) and real numbers (λ_{i_n}) with $n \in \bar{N} = \{1, \dots, \bar{N}\}$ such that for every $n \in \bar{N}$, $i_n \in I$, $\lambda_{i_n} > 0$, $x_{i_n} \in P_{i_n}(\bar{x}, \bar{y})$,

$$\zeta_{i_n} \in \left(\bar{x}_{i_n} + \sum_{j \in J} \theta_{i_n j}(Y_j - \bar{y}_j) \right) \cup \left(\omega_{i_n} + \sum_{j \in J} \theta_{i_n j} Y_j \right),$$

$\sum_{n \in \bar{N}} \lambda_{i_n} = 1$ and $\sum_{n \in \bar{N}} \lambda_{i_n}(x_{i_n} - \zeta_{i_n}) = 0$.

For every $i \in I$ set

$$\lambda_i = \sum_{\{n \in \bar{N} | i_n = i\}} \lambda_{i_n}, \qquad \eta_i = \sum_{\left\{n \in \bar{N} | i_n = i, \zeta_{i_n} \in \left(\omega_i + \sum_{j \in J} \theta_{ij} Y_j\right)\right\}} \lambda_{i_n}, \qquad \mu_i = \lambda_i - \eta_i.$$

For all $i \in I$ such that $\lambda_i > 0$, set

$$x_i = \frac{1}{\lambda_i} \sum_{\{n \in \bar{N} | i_n = i\}} \lambda_{i_n} x_{i_n}.$$

For all $i \in I$ such that $\eta_i > 0$ set

$$y_i = \frac{1}{\eta_i} \sum_{\left\{n \in \bar{N} | i_n = i, \zeta_{i_n} \in \left(\omega_i + \sum_{j \in J} \theta_{ij} Y_j\right)\right\}} \lambda_{i_n} \zeta_{i_n}.$$

[6] Without this assumption we could have the same problems as evoked in Example 8. An even weaker form of profit maximization would be needed. One has an incentive to change the production plan to a more profitable one if it allows also to buy a preferred point. If the preferred point is far away, it may be pointless to maximize profit.

For all $i \in I$ such that $\mu_i > 0$ set

$$z_i = \frac{1}{\mu_i} \sum_{\left\{n \in \bar{N} | i_n = i, \zeta_{i_n} \notin \left(\omega_i + \sum_{j \in J} \theta_{ij} Y_j\right)\right\}} \lambda_{i_n} \zeta_{i_n}.$$

For $i \in I$ such that $\lambda_i = 0$ choose an arbitrary $x_i \in X_i$, for all $i \in I$ such that $\eta_i = 0$ choose an arbitrary $y_i \in \left(\omega_i + \sum_{j \in J} \theta_{ij} Y_j\right)$ and for all $i \in I$ such that $\mu_i = 0$ choose an arbitrary $z_i \in \left(\bar{x}_i + \sum_{j \in J} \theta_{ij} (Y_j - \bar{y}_j)\right)$.

Thus,

$$\sum_{i \in I} \lambda_i x_i = \sum_{i \in I} \mu_i z_i + \sum_{i \in I} \eta_i y_i.$$

By the convexity of the production sets,

$$\sum_{i \in I} \mu_i z_i + \sum_{i \in I} \eta_i y_i \in \sum_{i \in I} \mu_i \left(\bar{x}_i + \sum_{j \in J} \theta_{ij} (Y_j - \bar{y}_j)\right) + \sum_{i \in I} \eta_i \left(\omega_i + \sum_{j \in J} \theta_{ij} Y_j\right).$$

For every $i \in I$ with $\lambda_i > 0$, $x_i \in P_i(\bar{x}, \bar{y})$ by the convexity of $P_i(\bar{x}, \bar{y})$. This contradicts that $(\bar{x}, \bar{y})$ is in the fuzzy rejective core. This ends the proof of the claim. □

Now, by iteratively applying the separating hyperplane theorem, we can choose a sequence of two by two orthogonal vectors $\{p^1, \dots, p^L\} \subset R^L \setminus \{0\}$ such that for every $r \in \{1, \dots, L\}$ and every $z \in \mathcal{K}^{r-1}$, $p^r \cdot z \geq 0$ where $\mathcal{K}^0 = \mathcal{K}$ and for $r \geq 1$, $\mathcal{K}^r = \mathcal{K} \cap \{p^1, \dots, p^r\}^{\perp}$.

For $i \in I$, let $\mathcal{K}_i = G_i \cup H_i$, $\mathcal{K}_i^0 = \mathcal{K}_i$, for $r \in \{1, \dots, L\}$, $\mathcal{K}_i^r = \mathcal{K}_i \cap \{p^1, \dots, p^r\}^{\perp}$. Note that $(\bar{x}, \bar{y}) \in \mathcal{FRC}$ implies that for all $i \in I$, $0 \notin \mathcal{K}_i$ and therefore $\mathcal{K}_i^L = \emptyset$. For every $j \in J$, let $r_j = \min r \in \{0, \dots, L\}$ such that for all $i \in I$ with $\theta_{ij} > 0$, $\mathcal{K}_i^r = \emptyset$.

Let $k = \min r \in \{1, \dots, L\}$ such that for all $i \in I$, $\mathcal{K}_i^r = \emptyset$. Set $\mathcal{P} = \{p^1, \dots, p^k\}$.

Claim 5. *For every $j \in J$, $\bar{y}_j \in S_j(\mathcal{P}(r_j))$.*

Proof. Suppose there exists $j' \in J$ such that $\bar{y}_{j'} \notin S_{j'}(\mathcal{P}(r_{j'}))$ and let $\bar{r}$ be the smallest r such that $\bar{y}_{j'} \notin S_{j'}(\mathcal{P}(\bar{r}))$. Choose $i \in I$ such that $\theta_{ij'} > 0$ and such that $\mathcal{K}_i^{\bar{r}} \neq \emptyset$. Thus, there exists $x_i \in P_i(\bar{x}, \bar{y})$ and $y \in \prod_{j \in J} Y_j$ with $\mathcal{P}(\bar{r}) y_{j'} > \mathcal{P}(\bar{r}) \bar{y}_{j'}$ and for $j \neq j'$, $y_j = \bar{y}_j$ such that

$$\mathcal{P}(\bar{r} - 1) x_i - \max \mathcal{P}(\bar{r} - 1) \left\{ \left(\omega_i + \sum_{j \in J} \theta_{ij} y_j\right), (\bar{x}_i + (y_{j'} - \bar{y}_{j'}) \right\} = 0.$$

Thus, for $\lambda \in]0, 1]$ small enough

$$\mathcal{P}(\bar{r})(\lambda x_i + (1 - \lambda) \bar{x}_i - \max \mathcal{P}(\bar{r}) \left\{ \left(\omega_i + \sum_{j \in J} \theta_{ij} y_j\right), (\bar{x}_i + y_{j'} - \bar{y}_{j'}) \right\} < 0.$$

This contradicts the iterative separation argument. □

For every $i \in I$, let $w_i = \max\{\mathcal{P}\bar{x}_i, \mathcal{P}\omega_i + \sum_{j \in J}\theta_{ij}\mathcal{P}\bar{y}_j\}$. Thus, $\bar{x}_i \in B_i(\mathcal{P}, w_i)$.

Claim 6. *For every $i \in I$, $B_i(\mathcal{P}, w_i) \cap P_i(\bar{x}, \bar{y}) = \emptyset$.*

Proof. The convexity of X_i implies by Lemma 2, $B_i(\mathcal{P}, w) = \overline{\{x_i \in X_i | \mathcal{P}x_i \leq w\}}$. If there exists $i \in I$ and $\xi_i \in B_i(\mathcal{P}, w_i) \cap P_i(\bar{x}, \bar{y})$, then there exists a sequence $\{\xi_i^n\} \subset X_i$ converging to ξ_i such that for all n, $\mathcal{P}\xi_i^n \leq w_i$. Since $P_i(\bar{x}, \bar{y})$ is open in X_i, for all large enough n, $\xi_i^n \in P_i(\bar{x}, \bar{y})$, but this contradicts the iterative separation argument. □

Given $(\bar{x}, \bar{y}, \mathcal{P}, w)$, one easily checks that for every $j \in J$, $r_j = \kappa_j$. Thus, $(\bar{x}, \bar{y}, \mathcal{P}, w)$ is a weak hierarchic equilibrium.

Corollary 2. *Suppose for every $i \in I$, X_i is convex, P_i has convex, open values in X_i and $J = \emptyset$. Then, the set of hierarchic equilibrium allocations and the set of fuzzy rejective core allocations coincide.*

This is immediate by the two previous propositions and the fact that in an exchange economy the weak hierarchic equilibrium and the hierarchic equilibrium coincides. In view of the remark after Definition 8, when minimum-wealth situations can be excluded, then the above propositions hold for a sort of dividend equilibrium where the profit maximizing firms are those whose shareholders are not all satiated in their preferences. If furthermore non-satiation holds, then the equivalence between Walras equilibria and the fuzzy rejective core is a corollary of Propositions 6 and 7.

Note that the non-emptiness of the fuzzy rejective core without an interiority condition follows from Theorem 1 together with Proposition 6.

10. Monotonicity

In the following proposition, we will prove that there exist hierarchic equilibria satisfying a monotonicity property. Monotonicity means that if agent alpha can propose all net trades beta can propose, then alpha will obtain dividends at least as high as beta (cf. Aumann and Drèze, 1986). This implies in particular that there always exists an equal treatment hierarchic equilibrium.

Proposition 8. *For every economy $\mathcal{E}$ satisfying* Assumptions 1–5 *there exists a hierarchic equilibrium $(x, y, \mathcal{P}, w)$, such that for all $i, i' \in I$, $X_i + t \subset X_{i'}$ and $\{\omega_i\} + \sum_{j \in J}\theta_{ij}Y_j + t \subset \{\omega_{i'}\} + \sum_{j \in J}\theta_{i'j}Y_j$ for some $t \in R^L$ implies $B_i(\mathcal{P}, w_i) + t \subset B_{i'}(\mathcal{P}, w_{i'})$.*

Proof. The proof consists in adding some arguments to the proof of Theorem 1. Suppose $\delta_1 = \cdots = \delta_I = 1$ and consider the sequence $B_i(p^n)$ converging to B_i. Suppose $x_i \in B_i$, then there exists a sequence x_i^n converging to x_i, such that for every n, $x_i^n \in B_i(p^n)$. Thus, for every n, $p^n \cdot (x_i^n + t) \leq p^n \cdot \left(\omega_i + \sum_{j \in J}\theta_{ij}\tilde{S}_j(p^n)\right) + p^n \cdot t + 1 - \|p^n\|$. One easily checks that the right-hand side is smaller than or equal to $p^n \cdot \left(\omega_{i'} + \sum_{j\in J}\theta_{i'j}\tilde{S}_j(p^n)\right) + 1 - \|p^n\|$. So for every n, $x_i^n + t \in B_{i'}(p^n)$ and the sequence converges to $x_i + t$, which is therefore in $B_{i'}$. □

11. Related generalized equilibrium concepts

The idea of hierarchic prices has been introduced in Gay (1978). He proposed generalized prices with two by two disjoint support, calling them "exchange rates". We gave the definition in Section 3. Based on such a price notion Danilov and Sotskov (1984, 1990) proposed equilibrium concepts for exchange and for production economies with consumption sets corresponding to the positive orthant. Mertens (1996) also proposes a special case of Danilov and Sotskov's (1990) equilibrium concept for linear exchange economies (cf. Florig, 1998). His concept is however designed to meet the requirements of some market mechanism rather then to generalize the Walras equilibrium in a broad setting.

Marakulin (1990) proposed a generalized equilibrium concept for exchange economies within terms of non-standard analysis. Existence is shown with convex consumption sets. He shows that his equilibrium concept can be reformulated in terms of standard analysis provided the consumption sets are polyhedral. In this case his equilibrium concept coincides with the hierarchic equilibrium.

The remaining part of this section is devoted to a comparison between Danilov and Sotskov's (1990) generalized equilibrium concepts and the hierarchic equilibrium.

Definition 9. An *exchange value* $\mathcal{P} = \{p^1, \ldots, p^k\}$ is an ordered family of vectors of R^L_+ such that for all $r, r' \in \{1, \ldots, k\}$, supp $p^r \neq \emptyset$, $r \neq r'$ implies supp $p^{r'} \cap$ supp $p^r = \emptyset$ and $\cup_{\rho=0}^k$ supp $p^\rho = L$.

An exchange value is a special case of exchange rate. We note $\mathcal{EV}$ the set of exchange values. Of course $\mathcal{EV} \subset \mathcal{HP}$.

Definition 10. A *PV-equilibrium* of an economy $\mathcal{E}$ is a collection $(x, y, \mathcal{P}, w) \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j \times \mathcal{EV} \times (\bar{R}^k)^I$ such that:

1. for all $i \in I$, $x_i \in B_i(\mathcal{P}, w_i)$ and $P_i(x, y) \cap B_i(\mathcal{P}, w_i) = \emptyset$;
2. for all $i \in I$, $\mathcal{P}\omega_i + \sum_{j\in J}\theta_{ij}\pi_j(\mathcal{P}) \leq w_i$;
3. $\Sigma_{i\in I} x_i \leq \Sigma_{j\in J} y_j + \omega$.

A major drawback of this concept is of course that no decision criterion is imposed on the firms. So it is not clear how the production plans are determined. We will see that another drawback is possible non-existence under standard assumptions. Danilov and Sotskov (1990) proposed the PV-equilibrium for the case $|I| = |J|$, for every $i \in I$, $X_i = R^L_+$ and $\theta_{ii} = 1$. For a certain very restrictive class of economies, one may prove existence of a PV-equilibrium (Florig, 1997), even with profit maximization such as in Section 2. However, a PV-equilibrium may fail to exist even under Danilov and Sotskov's (1990) assumptions.

Example 9. Consider an economy with three commodities, two consumers and a firm entirely belonging to consumer two. Let $X_1 = X_2 = R^3_+, u_1(x) = x^1, u_2(x) = x^1+x^2+x^3$, $\omega_1 = (1, 1, 1)$, $\omega_2 = (0, 0, 0)$, $Y_2 = \{y \in R^3 | y \le t(1, 1, -1), t \ge 0\}$ and $\theta_{22} = 1$. It is easy to check that no Walras equilibrium exists. We must have supp $p^1 = \{1, 3\}$. Clearly, commodity one should be in supp p^1, since otherwise consumer one will be able to buy it for free as he has strictly positive income with respect to any p^1. When commodity one is in supp p^1, commodity three must also be in supp p^1, since otherwise the firm can make an infinite profit with respect to p^1. Then by Definition 10(2), consumer two would have an infinite income on sub-market one and, given his preferences, Definition 10(1) could not hold. With commodity two in supp p^1, we would be in the standard case, thus, as no Walras equilibrium exists, 2 is in supp p^2. The unique exchange value, which may be candidate for a PV-equilibrium price, is $\mathcal{P} = \{(1, 0, 1), (0, 1, 0)\}$. However, $\pi_1(\mathcal{P}) = (0, +\infty)$, $B_2(\mathcal{P}, (0, +\infty)) = \{0\} \times R_+$ and hence for all $w_2 \in \bar{R}^2$ with $\mathcal{P}\omega_2 + \pi_2(\mathcal{P}) \le w_2$, $\text{argmax}_{x \in B_2(\mathcal{P}, w_2)} u_2(x) = \emptyset$. Hence, no PV-equilibrium exists for this economy which fits into Danilov and Sotskov's set up. A hierarchic equilibrium of this economy, is for example $(x, y, \{p^1, p^2\}, w)$ with $x_1 = (2, 1, 0)$, $x_2 = (0, 1, 0)$, $y_1 = (1, 1, -1)$, $p^1 = (1, 0, 1)$, $p^2 = (-1, 2, 1)$, $w_1 = (2, +\infty)$, $w_2 = (0, 2)$.

The generalized equilibrium concept Danilov and Sotskov (1990) proposed for exchange economies is called EV-equilibrium. The difference with the PV-equilibrium is that here one requires that for all $i \in I$, $P\omega_i = w_i$.

Definition 11. An *EV-equilibrium* of an exchange economy $\mathcal{E}$ is a collection $(x, \mathcal{P}) \in \prod_{i \in I} X_i \times \mathcal{EV}$ such that:

1. for all $i \in I$, $x_i \in B_i(\mathcal{P}, \mathcal{P}\omega_i)$ and $P_i(x) \cap B_i(\mathcal{P}, \mathcal{P}\omega_i) = \emptyset$;
2. $\Sigma_{i \in I} x_i \le \omega$.

Danilov and Sotskov (1990) proved existence of an EV-equilibrium in the case where for every $i \in I$, $X_i = R^L_+$. In the same framework, without free disposal, an EV-equilibrium is a hierarchic equilibrium (cf. Section 3). A Walras equilibrium is not necessarily an EV-equilibrium as defined (cf. Example 1; Florig, 1998). In Example 4 the consumption sets are different from the positive orthant and no EV-equilibrium exists. Another drawback of the EV-equilibrium is that its equilibrium allocation set is unstable, with respect to modifications of the economy which seem to be irrelevant to us. To see this, consider again Example 2.

Example 10. Let $X_1 = X_2 = R^2_+$, $\omega_1 = (1, 1)$, $\omega_2 = (0, 1)$ and $u_1(x) = x^1$, $u_2(x) = x^2$. Here, the EV-equilibria correspond to the worst hierarchic equilibria in terms of Pareto optimality: $(x, \{p^1, p^2\})$ with $x_1 = (1, t)$, $x_2 = \omega_2$ for $t \in [0, 1]$ and $p^1 = (1, 0)$, $p^2 = (0, 1)$. However, $(\xi, \{p^1, p^2\}, w)$ with $\xi_1 = (1, 0)$, $\xi_2 = (0, 2)$ and $w_1 = (1, +\infty)$, $w_2 = (0, 2)$ is also a hierarchic equilibrium.

Introduce a third commodity into the economy, which for both consumers is completely useless, but both own one unit of this commodity. This could be interpreted as paper money as in Kajii (1996). The "new" economy is now basically the same: $X'_1 = X'_2 = R^3_+$,

$\omega'_1 = (1, 1, 1), \omega'_2 = (0, 1, 1), u'_1(x) = x^1, u'_2(x) = x^2$. We now have, amongst others, an EV-equilibrium with $q^1 = (1, 0, 0)$, $q^2 = (0, 1, 1)$, $w_1 = (1, +\infty)$, $w_2 = (0, 2)$ and $x'_1 = (1, 0, 2), x'_2 = (0, 2, 0)$.

Note that the Walras equilibrium suffers from the same defect. Consider again Example 6.

Example 11. Let $X_1 = X_2 = R_+, \omega_1 = \omega_2 = 2, u_1(x) = x$,

$$u_2(x) = \begin{cases} 0, & \text{if } x \leq 3, \\ x - 3, & \text{if } x \geq 3. \end{cases}$$

This example satisfies all assumptions needed to ensure existence as in Debreu (1962). There is a unique Walras equilibrium $x_1 = 2, x_2 = 2, p = 1$. Now introduce a second commodity playing the role of paper money. Let $\omega_1 = (2, 2), \omega_2 = (2, t)$ for $t \in [0, 1]$. Then, $x'_1 = (4, 0), x'_2 = (0, 2 + t), p' = (1, 1)$ is a new Walras equilibrium which Pareto dominates the first.

It is easy to see that the dividend equilibrium does not share this defect since the value of supplementary commodities which do not enter the preferences are simply transferred into the slack variable (cf. Kajii, 1996). The same holds for hierarchic equilibria. Some additional revenue of new and useless commodities just enter the dividend structure. So the set of hierarchic equilibria is not changed by such modifications.

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