Incomplete Markets with Nominal Assets: Slack and Money*

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Abstract

In this paper, we provide an equilibrium analysis in the framework of incomplete markets where some agents' preferences are possibly satiated at some state of the nature. We will consider nominal assets with exogenously fixed asset prices. We extend the notion of equilibrium with slack - introduced by Drèze and Müller [4] in a fixed price setting - to the GEI framework.

Unlike Cass [2], our existence prove does not break the symmetry of the problem. This has also the advantage to extend to the case of large economies.

Keywords: incomplete markets, nominal assets, satiated preferences, equilibrium with slack, paper money, large economy.

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1 Introduction

In the general equilibrium model with incomplete markets, the existence of equilibria relies on the assumption that investors are non-satiated at each state of the nature. This means that it is always possible for each state of the nature to change consumption for this state, keeping it fixed for the other states, and increase thereby utility.

In the Capital Asset Pricing Model (CAPM), the utility functions are in general assumed to depend only on the mean and the variance of the wealth associated to a portfolio. This is true if the utility is of the von Neumann - Morgenstern type with a quadratic utility function or if the revenue is normally distributed. One easily checks that this type of utility functions does not satisfy the non-satiation property at each state of the nature. The necessary condition derived in the CAPM model for the existence of an equilibrium situation may not be sufficient since the equilibrium existence results in financial economies require the non-satiation property of the preferences. Furthermore, the restriction of the analysis to normal distributions is problematic: the consumption or the revenue is determined at the equilibrium and thus is endogenous and may not be normally distributed; moreover, this ignores completely the case of finite or discrete distributions which occurs frequently in practice.

In the complete market setting an equilibrium may fail to exist without the non-satiation property. At any given price some of the consumers may want to consume in the interior of his consumption set. Thus, Walrs' law and therefore equilibrium existence fails. Fixed price setting (Drèze and Müller [4]) and the absence of the strong survival assumption may lead to the same problems, even if preferences are locally non-satiated.

The existence problem can be overcome by allowing some agents to spend more than the value of their initial endowments. Such an equilibrium is called dividend equilibrium or equilibrium with slack. It was first introduced in a fixed price setting in Drèze and Müller [4] and this was later adapted to the standard model of complete markets by Makarov [11], Aumann and Drèze [1] and Mas-Colell [12]. As pointed out by Kahji [10] the slack in consumers income may be interpreted as the value of paper money which is held by the consumers as initial endowment. The value of this paper money may then be positive if non-satiation fails to hold.

We will adapt the notion of equilibrium with slack and equilibrium with money to the case of incomplete markets. Some slack in the income will be allowed at each state of the nature. This extra income may be interpreted as the value of commodities which are not explicitly taken into account in the model since irrelevant for consumption.

Indeed, in the case of satiation of the preferences, an equilibrium may still exist, but as shown by the following example, with slack or paper money new equilibrium allocations occur.

<u>Example</u> 1 There are two periods $t = 0$ and $t = 1$ and one physical commodity per period, no financial asset is available which would allow wealth transfer across states. The two consumers are respectively endowed with $e_1 = (3, 1)$ and $e_2 = (1, 3)$. Their preferences exhibit strong complementarity between present and future consumption represented by the utility functions $u_1(x) = u_2(x) = \min\{x_1, x_2\}$. There are three spot-equilibria : $x_i = e_i$ for $i = 1, 2$ with $p \gg 0$; $x_1 = (3, 3)$, $x_2 = (1, 1)$ with $p = (1, 0)$ and $x_1 = (1, 1)$, $x_2 = (3, 3)$ with $p = (0, 1)$. Now introduce for each state a second commodity and allocate one unit to each consumer at each date. Thus, $\tilde{e}_1 = (3, 1, 1, 1)$, $\tilde{e}_2 = (1, 1, 3, 1)$. The preferences do not depend on the amount of the new good held by the agents. The set of spot-equilibria contains the previous equilibria with each agent keeping his endowment in the new commodity and the price of the new commodity set equal to zero. However the following is now also an equilibrium: $x_1 = (2, 2, 2, 0)$, $x_2 = (2, 0, 2, 2)$ with $p = (1, 1, 1, 1)$. The new equilibrium allocation improves the utility of both consumers as opposed to the equilibria without the new commodity. So the equilibrium set is perturbed by the introduction of useless goods. We will see that this is not the case if one considers equilibria with slack.

We could attain the same equilibrium allocation by alternatively introducing a durable good in period 0 which we call paper money. Equally this commodity does not enter the preferences. Give a unit to each consumer in period 0. Then in period 0 consumer 1 buys the unit of paper money of consumer 2 with price of money being 1 and in period 1, consumer 1 sells both units of paper money he now holds to consumer 1 at price 1/2. Equilibrium allocations are thus $x_1 = (2, 2, 2, 0)$, $x_2 = (2, 0, 2, 2)$ with $p = (1, 1, 1, 1/2)$.

In the present paper we will consider nominal assets, so for example non-indexed bonds. The rank of asset return matrix is thus continuous with respect to prices. In the numéraire or real asset case continuity can only be achieved by imposing restrictions on the preferences to be considered which are not satisfied for instance by the above example (Geanakoplos and Polemarchakis [8], Duffie and Shafer [5]). With nominal assets, the equilibrium analysis is usually on the Cass-trick (Cass [2]) by singling out a consumer and giving him an Arrow-Debreu budget set as in the complete market case. Then, a fixed point argument establishes the existence of a quasi-equilibrium where this special consumer obtains a maximal element of his budget set. This quasi-equilibrium is a financial equilibrium provided the price vector is non-zero in every state. The non-satiation of this consumer state by state implies that the price vector must be non-zero in every state.

The Cass trick allows for proving existence for any fixed arbitrage free asset price vector, whereas, Werner [15] gives a (symmetric) proof for the existence of an equilibrium for some endogeneous asset price vector. From a technical point view, the Cass trick breaks the symmetry of the argument and without the non-satiation property, it is not working through anymore since the Arrow-Debreu consumer may very well be satiated in some states and there, the quasi-equilibrium price could be zero. In the case of economies with a continuum of consumers, every consumer is negligible. Thus it becomes pointless to single out a consumer and the Cass trick cannot be applied.

We provide a symmetric existence argument for a nominal financial structure.[1] As in Cass ([2]) we fix asset prices exogeneously to an arbitrary arbitrage free asset price vector. We apply this argument on the one hand in order to prove the existence of a financial equilibrium with slack without non-satiation. On the other hand, we outline an adaptation of our argument for an existence proof in the case of a continuum of traders. Furthermore we study a notion of equilibrium with paper money and its relation to equilibria with slack.

2 Equilibria with Slack

There are two periods $t = 0$ and $t = 1$. Symmetric uncertainty concerns the second period, where, one state $s \in \{1, \dots, S\}$ out of a finite number of states of the nature occurs. We note $\mathbb{S} = \{0, \dots, S\}$, where $s = 0$ denotes the first period. There is a finite set $L = \{1, \dots, \ell\}$ of physical goods, available at each state of the nature. The commodity space is $\mathbb{R}^{\ell(S+1)}$. There is a finite set $I = \{1, \dots, I\}$ of households, each is characterized by a consumption set $X_i \subset \mathbb{R}^{\ell(S+1)}$, a preference correspondence $P_i : X \to X_i$, where $X = \prod_{i \in I} X_i$ is the aggregate consumption set, and an initial vector of endowments $e_i = (e_{i0}, ..., e_{iS}) \in \mathbb{R}^{\ell(S+1)}$.

We denote X_{is} the projection of X_i on the s-th component. Denote the set of feasible allocations by

$$\mathcal{A} = \left\{ x \in \prod_{i \in I} X_i \;\middle|\; \sum_{i \in I} (x_i - e_i) = 0 \right\}.$$

The financial market is composed of a finite number J of assets traded at $t = 0$.

[1] Note that in the case of real assets Polemarchakis and Siconolfi [14] gave a symmetric existence proof.

The returns are described by a $(S \times J)$-matrix

$$V = (r^j(s))_{\substack{s=1,\dots,S \\ j=1,\dots,J}},$$

where the vector $(r^j(s))_{s=1}^S \in \mathbb{R}^S$ describes the random return of asset j expressed in terms of units of account. Each consumer will choose a portfolio $z_i \in \mathbb{R}^J$ allowing him to make wealth transfers across the states of nature. Since we will not assume non-satiation of the preferences of the consumers, we denote $d_i = (d_{is}) \in \mathbb{R}_+^{S+1}$ the vector of slack for consumer i. The budget set of a consumer in this setting is defined as follows

$$B_i^F(p,q,d_i) = \left\{ x \in X_i \;\middle|\; \begin{array}{l} \exists z_i \in \mathbb{R}^J \\ p \,\square\, (x - e_i) \leq W\, z_i + d_i \end{array} \right\}$$

where W is the $(S+1) \times J$ matrix : $\begin{pmatrix} -q \\ V \end{pmatrix}$. Remark that if $d_i = 0$, then $B_i^F(p,q,d_i)$ coincides with the usual budget set.

<u>Definition</u> 2.1 (Equilibrium with Slack in Incomplete Markets) An element (x, z, d, p, q) of $X \times \mathbb{R}^{JI} \times \mathbb{R}_+^{(S+1)I} \times \mathbb{R}^{L(S+1)} \times \mathbb{R}^J$, is a *equilibrium with slack* if :

- For $i = 1, ..., I$, $p \,\square\, (x_i - e_i) \leq W\, z_i + d_i$ and $P_i(x) \bigcap B_i^F(p,q,d_i) = \emptyset$;
- $\displaystyle\sum_{i \in I}(x_i - e_i) = 0$ and $\displaystyle\sum_{i \in I} z_i = 0$.

It is easy to check that this equilibrium notion is stable with respect to the introduction of goods which are irrelevant for consumption as in Example 1. It is indeed sufficient to shift the extra income from these goods into the slack variable.

<u>Example</u> 2 We first give an example where an equilibrium does not exist. Consider an economy with one physical commodity and three states of the nature, $\mathbb{S} = \{0, 1, 2\}$. There are no assets allowing to transfer wealth across the states of the nature. There are two households which consumption sets are given by $X_1 = X_2 = \{0\} \times \mathbb{R}_+^2$ meaning that there is no consumption at date $t = 0$. The utility functions and endowments are given by :

$$\begin{array}{ll} u_1(x) = x_1 - x_2 \qquad & e_1 = (1,1) \\ u_2(x) = x_1 + x_2 \qquad & e_2 = (1,1) \end{array}$$

One checks easily that this economy does not have any equilibrium and that it admits an equilibrium with slack given by :

$$\begin{array}{l} x_1 = (1,0), \quad x_2 = (1,2) \\ p_1 = p_2 = 1 \\ d_1 = d_2 = (0,1). \end{array}$$

<u>Assumption</u> C *For $i = 1, ..., I$:*

1. *X_i is closed, convex and bounded below;*

2. *$P_i : X \to X_i$ has an open graph in $X \times X_i$ with convex values and for all $x \in X$, $x \notin P_i(x)$.*

ASSUMPTION S (Survival Assumption) *For $i = 1, ..., I$, $e_i \in \text{int} X_i$.*

ASSUMPTION NS(s) For $i = 1, \dots, I$: for all $x \in \mathcal{A}$, $\forall \epsilon > 0$, there exists $\tilde{x}_i \in P_i(x) \bigcap \mathbb{B}(x, \epsilon)$ such that $\tilde{x}_{is'} = x_{is'}$ if $s' \neq s$.

PROPOSITION 2.1 *Let (x, z, d, p, q) be an equilibrium with slack. Then :*

1. *If NS(s) holds, then for $i = 1, \dots, I$, $d_{is} = 0$;*

2. *If NS(s) holds for $s = 0, \dots, S$ then $((x, z), p, q)$ is an equilibrium.*

Proof. From assumption NS(s), one has for all i, $p_s \cdot (x_{is} - e_{is}) = (W\, z_i)_s + d_{is}$. Summing on i we obtain, $\sum_{i \in I} d_{is} = 0$. Then for all $i \in I$, $d_{is} = 0$. □

An asset price q is called arbitrage-free, if there exists a vector of node prices $\beta \in \mathbb{R}^S_{++}$ such that $q = \beta V$. This property of the financial market implies that there is no portfolio $z \in \mathbb{R}^J$ which returns are non negative at each state of the nature, i.e., $Wz \geq 0$ and $Wz \neq 0$. Such an arbitrage opportunity, is ruled out, at the equilibrium if the preferences of the households are non-satiated at each state of the nature. The redundancy of the financial market clearing condition ($\sum_{i=1}^{I} z_i = 0$) is in fact a consequence of this property. If the preferences are possibly satiated, the free arbitrage property of the asset prices is no more a necessary condition at equilibrium. This is illustrated in the following example.

EXAMPLE 3 There are two dates $t = 0, 1$, 4 states of the nature at $t = 1$ and one commodity per state of the nature. The economy is composed of two consumers $i = 1, 2$:

$$\begin{aligned} &X_i = \{0\} \times \mathbb{R}^4_+ \\ &e_1 = e_2 = (0, 1, 1, 1, 1) \\ &u_1(x) \;=\; x_1 - x_4 + \min\{1, x_2\} + \min\{1, x_3\} \\ &u_2(x) \;=\; -x_1 + x_4 + \min\{1, x_2\} + \min\{1, x_3\} \end{aligned}$$

The financial market is composed of two nominal assets which returns are given by the vectors $a = (1, 1, 0, -1)$ and $b = (-1, 0, 1, 1)$. Remark that the asset price $q = (0, 0)$ is not arbitrage free. Let $(x_1, x_2, z_1, z_2, p, q)$ satisfy all equilibrium conditions, but not necessarily $z_1 + z_2 = 0$. Then, by the arbitrage possibility and feasibility, we must have $x_{12} = x_{13} = x_{22} = x_{23} = 1$. Moreover, $x_{11} = 1 + p_4/p_1$, $x_{14} = 0$ and $x_{21} = 0$, $x_{24} = 1 + p_1/p_4$. Thus $p_1 = p_4 = 1$. We may assume without any loss of generality

$p_0 = 1$. So $x_1 = (0, 2, 1, 1, 0)$, $x_2 = (0, 0, 1, 1, 2)$, $p = (1, 1, p_2, p_3, 1)$. Since in state 2 and 3 both keep their initial endowment, we have $Vz_1 = (1, t_1, \tau_1, -1)$ with $(t_1, \tau_1) \geq 0$ and $Vz_2 = (-1, t_2, \tau_2, 1)$ with $(t_2, \tau_2) \geq 0$. Thus $z_1 = \lambda a + \mu b \neq 0$ with $(\lambda, \mu) \geq 0$ with at least one different from zero. Suppose asset markets clear, i.e. $z_2 = -z_1$, then consumer 2 has a negative return in state 2 if $\lambda > 0$ and in state 3 if $\mu > 0$. A contradiction. With $p_2 = p_3 = 1$ and $z_1 = (1, 0)$ and $z_2 = (0, 1)$ we have an equilibrium on the goods market.

In the following, we show that the notion of equilibrium with slack coincides with the usual concept when markets are complete.

DEFINITION 2.2 (Equilibrium with Slack in Complete Markets) An element (x, δ, P) of $X \times \mathbb{R}^I_+ \times \mathbb{R}^L$, is an *equilibrium with slack* if :

- For $i = 1, ..., I$, $x_i \in B_i^{AD}(P, \delta_i) = \{y_i \in X_i \mid P \cdot (y_i - e_i) \leq \delta_i\}$ and

$$P_i(x) \bigcap B_i^{AD}(P, \delta_i) = \emptyset;$$

- $\displaystyle\sum_{i \in I}(x_i - e_i) = 0.$

PROPOSITION 2.2 *Assume that financial market structure is complete (i.e.* rank $V =$ *S). Then :*

1. *Let (x, z, d, p, q) be an equilibrium with slack such that q is arbitrage free and $\lambda \in \mathbb{R}^{S+1}_{++}$ the vector of the associated node prices. Then $(x, (\lambda \cdot d_i), (\lambda_s p_s))$ is an equilibrium with slack.*

2. *Let (x, δ, P) an equilibrium with slack. Then for any arbitrage free asset price q, there exist (p, d, z) such that (x, z, d, p, q) is an equilibrium with slack.*

Proof. The proof of point 1 is a consequence of the equivalence of the Arrow-Debreu budget set with the budget set with financial market. For point 2, let $q \in \mathbb{R}^J$ be an arbitrage free asset price vector and $\lambda \in \mathbb{R}^{S+1}_{++}$ the vector of associated node prices. Let $p \in \mathbb{R}^{\ell(S+1)}$ such that for all $s \in \mathbb{S}$, $P_s = \lambda_s p_s$ and for all i let $d_i \in \mathbb{R}^{S+1}$ arbitrarily chosen in $\left\{\gamma \in \mathbb{R}^{S+1}_+ \mid \lambda \cdot \gamma = \delta_i\right\}$. Since Markets are complete, one easily checks that $B_i^{AD}(P, \delta_i)$ coincides with $B_i^F(p, q, d_i)$. The fact that the obtained portfolios (z_i) satisfy $\sum_{i=1}^I z_i = 0$ is a consequence of the fact that q is arbitrage free. □

THEOREM 2.1 *Under Assumptions C, S, for every arbitrage free price vector $q \in \mathbb{R}^J$ there exists an equilibrium with slack (x, z, d, p, q) such that, for all $i, i' \in I$, $d_i = d_{i'}$.*

COROLLARY 2.1 *If Assumptions C, S, and for all $s \in \mathbb{S}$, NS(s) holds, then, for every arbitrage free $q \in \mathbb{R}^J$ there exists an equilibrium (x, z, p, q).*

In our examples, we assumed for convenience that no consumption takes place at period 0. Then of course assumption S cannot hold. However, Theorem 2.1 remains valid assuming interiority of the initial endowments for each state where consumption takes place. The proof would need to be changed mostly in Claim 5.11. We omit this since complexity of notations would increase.

When no consumption of commodities is possible at $t = 0$, also NS(0) fails to hold. The existence of equilibria is problematic when financial structure is composed of numéraire assets as in Polemarchakis and Siconolfi [13] and Gottardi and Hens [9]. Satiation points according to the portfolios may appear even if the preferences satisfy the monotonicity assumption according to the commodities. The following example illustrates this point.

EXAMPLE 4 Consider an economy with two periods $t = 0, 1$ and two possible states of the nature at $t = 1$. There is one physical commodity. The two households are characterized by :

$$\begin{array}{ll} X_1 = X_2 = \{0\} \times \mathbb{R}^2_+ & \\ u_1(x) = 2x_1 + x_2 & e_1 = (1,1) \\ u_2(x) = x_1 + 2x_2 & e_2 = (2,1) \end{array}$$

The financial market is composed of one nominal asset promising the vector of returns $(1, -1)$ at $t = 1$. Remark that since the budget constraints are not homogeneous, one can not a priori normalize the commodity prices at each state of the nature. This economy admits equilibria for any asset price described as follows :

- $q < 0$, $x_1 = e_1$, $x_2 = e_2$,$z_1 = z_2 = 0$ and $p = (2, 1)$;
- $q = 0$, $x_1 = (2, 0)$, $x_2 = (1, 2)$,$z_1 = 1$, $z_2 = -1$ and $p = (1, 2)$;
- $q > 0$, $x_1 = e_1$, $x_2 = e_2$, $z_1 = z_2 = 0$ and $p = (1, 2)$.

But this economy admits also some equilibria with slack which are not equilibria :

- $q < 0$, $x_1 = (2, 0)$, $x_2 = (1, 2)$, $z_1 = 1$, $z_2 = -1$, $p = (1, 1)$ and $d_1 = d_2 = (-q, 0, 0)$.

Consider now the same economy with the difference that the asset is supposed to be a numéraire asset. Working with numéraire assets, one can normalize the commodity prices at each state of the nature since the budget constraints are homogeneous. This economy does not admit any equilibrium but has the same equilibrium with slack as above (cf. Polemarchakis and Siconolfi [13]).

3 Equilibria with money

We introduce an additional durable good (paper money) which is distributed in $t = 0$ and may be consumed at any non-negative amount. Preferences are independent of this good, it could however serve as a medium of exchange. The consumption set for this durable good is $\mathbb{R}_+^{S+1}$ for every consumer. Consumer i is initially endowed with M_{i0} units of this good. His demand for this good at time $t = 0$, m_{i0} constitutes his initial endowment of money at $t = 1$ for all $s \in S$, $M_{is} = m_{i0}$.

Thus, the economy $\mathcal{E}$ is defined by $((X_i, P_i, e_i, M_{i0})_{i=1,\dots,I}, F)$. The vector price of paper money will be denoted $\pi \in \mathbb{R}^{S+1}$. The budget set of a consumer is defined as follows : $B_i^F(p, \pi, q) =$

$$\left\{(x, m, z) \in X_i \times \mathbb{R}_+^{S+1} \times \mathbb{R}^J \mid p \,\square\, (x - e_i) + \pi \,\square\, (m_i - M_i) \leq W(p, q) z_i\right\},$$

where $M_i = M_i(m_i) = (M_{i0}, m_{i0}, \dots, m_{i0})$.

<u>Definition</u> 3.1 An element $((x_i, m_i, z_i)_i, p, \pi, q)$ of $X \times \mathbb{R}_+^{I(S+1)} \times \mathbb{R}^{JI} \times \mathbb{R}^{L(S+1)} \times \mathbb{R}^{S+1} \times \mathbb{R}^J$, is a *financial equilibrium (with money)* of $\mathcal{E}$ if :

- For $i \in I$, $(x_i, m_i, z_i) \in B_i^F(p, q)$ and
$$\left(P_i(x) \times \mathbb{R}_+^{S+1} \times \mathbb{R}^J\right) \bigcap B_i^F(p, \pi, q) = \emptyset;$$
- $\sum_{i=1}^{I}(x_i - e_i) = 0$, $\sum_{i=1}^{I}(m_i - M_i) = 0$ and $\sum_{i=1}^{I} z_i = 0$.

The following example shows that in general we cannot identify the set of equilibria with money with the set of equilibria with slack unlike in the case of complete markets (Kahji [10]).

<u>Example</u> 5 Consider an economy with two periods $t = 0, 1$ and only one state of the nature at $t = 1$. There is one physical commodity. The two households are characterized by :

$$\begin{array}{ll} X_1 = X_2 = \mathbb{R}_+^2 & \\ u_1(x) = \sqrt{x}_0 + \sqrt{x}_1 & e_1 = (0, 1) \\ u_2(x) = \min\{x_0, x_1\} & e_2 = (1, 2) \end{array}$$

There is no financial market. The allocation $x_1 = (0, 2), x_2 = (1, 1)$ can be supported by a dividend equilibrium with dividends $d_1 = d_2 = (0, 1)$ and spot prices $p = (1, 1)$. It can however not be supported as an equilibrium with money. We would need the price of money to be positive in period 1. Since consumer 1's utility is strictly increasing in period 1, the price of money must be strictly positive in period 0 since otherwise his demand for money would be infinite in 0. Moreover, if money is distributed only in period 0, his initial money endowment in period 0 must be strictly positive. Since his marginal utility for consumption in 0 is infinite he would spend some of the money to buy goods in 0 - a contradiction.

Now set the initial endowments to $e_1 = (3,1), e_2 = (1,3)$. The allocation $x_1 = (2,2), x_2 = (2,2)$ can easily be supported by an equilibrium with money ($M_0 = 1, M_1 = 0$, $m_1 = (1,0), m_2 = (0,1)$, $p = (1,1)$ and price of money $\pi = (1,1)$. It cannot be supported by dividend equilibrium. The dividend would need to enable consumer 1 to buy a unit in period 1, without having to give up commodities in period 0.

Theorem 3.1 *Assumptions C, S and B hold. Then, there exists a financial equilibrium with money.*

We do not give explicit proof of this result since it is an easy adaptation of standard analysis : one maximises state by state the aggregate excess demand (in zero the excess demand in goods and assets), for all positive integers n we choose $(p(0), q)$ and for all s, $p(s)$ in the closed ball of center 0 and radius $1 - 1/n$, choose the price of paper money as follows :

$$\pi_0 = \pi_0^n(p,q) = 1 - \|(p(0), q)\|$$

$$\pi_s^n(p,q) = 1 - \|p(s)\|, \ s \in S.$$

Finally we go to the limit decompactifying the consumption set and the set we choose the portfolio in. As the assets are nominal we have no discontinuity problem in the rank of the return matrix.

4 Large Economies

We outline the argument for the existence of an equilibrium in the case of a large economy with an incomplete nominal financial structure. The *continuum* of consumers is defined by a positive finite, complete measure space $(M, \mathcal{M}, \mu)$, where $\mathcal{M}$ is the σ-algebra of subsets in M, and μ a σ-additive positive measure on $\mathcal{M}$ with $\mu(M) = 1$. Each consumer a has a consumption set $X(a) \subset \mathbb{R}^{\ell(S+1)}$ and a preference relation, denoted $\preceq_a$, which is a complete preordering[2] defined on $X(a)$. The strict preference relation $\prec_a$ on $X(a)$ is defined by, for $x \in X(a)$, $x' \in X(a)$, $x \prec_a x'$ if [$x \preceq_a x'$ and not $x' \preceq_a x$]. For all $x \in X(a)$, let $P_a(x) = \{x' \in X(a) \mid x \prec_a x'\}$ be the set of consumption plans which are preferred to x.

[2]The binary relation $\preceq_a$ is reflexive, transitive and complete.

The set of consumption plans of this economy, corresponds to the set of integrable selections of the correspondence X, this set will be denoted by S^1_X. Formally:

$$S^1_X = \left\{ x(.) \in L^1(M, \mathbb{R}^{\ell(S+1)}) \mid x(a) \in X(a) \text{ a.e. } a \in M \right\}.$$

The initial endowments of consumer a are represented by a vector $e(a)$ of $\mathbb{R}^{\ell(S+1)}$. The function $e(.) : M \to \mathbb{R}^{\ell(S+1)}$ is assumed μ-integrable. Let $e = \int_M e(a) d\mu(a)$.

<u>Definition</u> 4.1 An element (x, z, p, q) of $S^1_X \times \mathcal{L}^1(M, \mathbb{R}^J) \times \mathbb{R}^{L(S+1)} \times \mathbb{R}^J$, is a equilibrium of the large economy if :

- for a.e. $a \in M$, $p \,\square\, (x(a) - e_i) \leq W\, z(a)$;
- for a.e. $a \in M$, $P_a(x(a)) \bigcap \left\{ x \in X(a) \,\middle|\, \begin{array}{l} \exists z_i \in \mathbb{R}^J \\ p \,\square\, (x - e(a)) \leq W\, z(a) \end{array} \right\} = \emptyset$;
- $\displaystyle\int_M x(a) d\mu(a) \leq e$ and $\displaystyle\int_M z(a) d\mu(a) = 0$.

In order to prove the existence of an equilibrium for the large economy, we state the following assumptions.

<u>Assumption</u> M

- The correspondence X from M to the subsets of $\mathbb{R}^{\ell(S+1)}$ is μ-measurable[3];
- The preference relation is measurable[4].

<u>Assumption</u> C'

- For a.e. $a \in M$, $X(a)$ is closed. The correspondence X is bounded below[5] and admits a selection $\hat{x}(.)$ such that[6] $\operatorname{ess\,sup}_{a \in M} \|\hat{x}(a)\| < \infty$;
- For a.e. $a \in M$, $\preceq_a$ is continuous[7];
- For every atom[8] C of M and a.e. $a \in C$, $X(a)$ is convex and $\preceq_a$ is convex[9].

[3]The graph $GX = \left\{ (a, x) \in M \times \mathbb{R}^{\ell(S+1)} \mid x \in X(a) \right\}$ belongs to $\mathcal{M} \otimes \mathcal{B}(\mathbb{R}^{\ell(S+1)})$, where $\mathcal{B}(\mathbb{R}^{\ell(S+1)})$ is the σ-algebra of Borel subsets of $\mathbb{R}^{\ell(S+1)}$.

[4]$\{(a, x, x') \in M \times \mathbb{R}^{\ell(S+1)} \times \mathbb{R}^{\ell(S+1)} \mid x \preceq_a x'\} \in M \otimes \mathcal{B}(\mathbb{R}^{\ell(S+1)}) \otimes \mathcal{B}(\mathbb{R}^{\ell(S+1)})$.

[5]there is a μ-integrable function $g : M \to \mathbb{R}^{\ell(S+1)}$ such that for a.e. $a \in M$, $g(a) \leq X(a)$.

[6]$\operatorname{ess\,sup}_{a \in M} \|\hat{x}(a)\| := \sup\{\alpha > 0 \mid \mu\{a \in M \mid \|\hat{x}(a)\| \geq \alpha\} > 0\}$.

[7]For all $x \in X(a)$, the sets $\{z \in X(a) \mid x \preceq_a z\}$ and $\{z \in X(a) \mid z \preceq_a x\}$ are closed.

[8]$C \in \mathcal{M}$ is called an atom of the measure space $(M, \mathcal{M}, \mu)$ if $\mu(C) \neq 0$ and $[B \in \mathcal{M}$ and $B \subset C]$ implies $\mu(B) = 0$ or $\mu(C \setminus B) = 0$.

[9]For every $x \in X(a)$, the set $\{z \in X(a) \mid x \preceq_a z\}$ is convex. Note that this assumption is always satisfied if the measure space M is atomless.

ASSUMPTION LNS For a.e. $a \in M$: for all $x \in X(a)$ such that $P_a(x) \neq \emptyset$, $x \in \overline{P_a(x)}$.

ASSUMPTION S' For a.e. $a \in M$, $e(a) \in \text{int}X(a)$.

THEOREM 4.1 *Suppose the large economy satisfies Assumptions M, C', LNS, S' then for every arbitrage free price $q \in \mathbb{R}^J$, there exists an equilibrium (x, z, p, q).*

5 Proof of Theorem 2.1.

Consider the augmented preferences (Gale and Mas Colell [6]) defined as follows :

$$\hat{P}_i(x) = \{(1-t)x_i + ty,\, t \in]0,1],\, y \in P_i(x)\}\,.$$

The correspondence $\hat{P}_i$ satisfies all the properties imposed by assumption C. Since for $i = 1, \ldots, I$, X_i is lower bounded and closed, $\mathcal{A}$ is compact. Thus, there exists $r > 0$, such that $\mathbb{B}(0, r)$ contains the projection of $\mathcal{A}$ on X_i in its interior. Let $\tilde{X}_i = X_i \cap \overline{\mathbb{B}(0, r)}$ for all $i \in I$. The openess of the graph of $\hat{P}_i$ in $\left(\prod_{j\in I} X_j\right) \times X_i$ implies the openess of its restriction to $\prod_{j\in I} \tilde{X}_j \times \tilde{X}_i$ denoted by $\tilde{P}_i$. Also irreflexivity and convex valuedness of $\tilde{P}_i$ follow from irreflexivity and convex valuedness of $\hat{P}_i$. Note $\tilde{X} = \prod_{i\in I} \tilde{X}_i$. Consider the sets of prices

$$\begin{array}{rcl} \mathbb{P}^n & = & \left\{p \in \mathbb{R}^{\ell(S+1)} \mid \|p\| \leq 1 - \frac{1}{n}\right\}; \\ \mathbb{P} & = & \left\{p \in \mathbb{R}^{\ell(S+1)} \mid \|p\| \leq 1\right\}. \end{array}$$

We fix an arbitrage free asset price vector $q \in \mathbb{R}^J$ and we assume without loss of generality that V is one-to-one. This implies that W is one-to-one. Let

$$\mathbb{Z}^n = \left\{z \in \mathbb{R}^J \mid \|z\| \leq n\right\},$$

and $\alpha(p) : \mathbb{P} \to \mathbb{R}^{S+1}$ be defined by :

$$\alpha_s(p) = \frac{1 - \|p\|}{I}.$$

For $i = 1, ..., I$, let $\varphi_i : \mathbb{P} \times \tilde{X} \times \mathbb{R}^{JI} \longrightarrow \mathbb{R}^{S+1}$ be defined by :

$$\varphi_i(p, x, z) = (I-1)\alpha(p) + \sum_{j\in I\setminus\{i\}} (W\, z_j - p \,\square\, (x_j - e_j)), \tag{1}$$

and let $\varphi : \mathbb{P} \times \tilde{X} \times \mathbb{R}^{JI} \longrightarrow \mathbb{R}^{S+1}$ be defined by

$$\varphi_s(p, x, z) = \max\{\varphi_{1s}(p, x, z), \ldots, \varphi_{Is}(p, x, z), 0\}, \quad \forall s \in \mathbb{S}.$$

For each consumer, $i = 1, \ldots, I$, consider $B_i(p, x, z) =$

$$\{(x'_i, z'_i) \in \tilde{X}_i \times \mathbb{Z}^n |\, p \,\square\, (x'_i - e_i) \leq W\, z'_i + \alpha(p) + \varphi(p, x, z)\}$$

For all $(p,x,z) \in \mathbb{P}^n \times \tilde{X} \times \mathbb{Z}^{nI}$, let

$$\forall i \in I, \quad \Phi_i^n(p,x,z) = \begin{cases} B_i(p,x,z) & \text{if } (x_i,z_i) \notin B_i(p,x,z), \\ B_i(p,x,z) \cap \tilde{P}_i(x) \times \mathbb{Z}^n & \text{otherwise,} \end{cases} \tag{2}$$

$$\Phi_0^n(p,x,z) = \left\{ p' \in \mathbb{P}^n \,\middle|\, p' \cdot \sum_{i\in I}(x_i - e_i) > p \cdot \sum_{i \in I}(x_i - e_i) \right\}. \tag{3}$$

<u>CLAIM</u> 5.1 *For all $n \geq 2$, for $i = 0,...,I$, the correspondence Φ_i^n is lower semi continuous with convex values.*

It is easy to see that Φ_0^n is continuous and convex valued. On $\mathbb{P}^n \times \tilde{X} \times \mathbb{Z}^{nI}$, for all $i \in I$ B_i is continuous and $\tilde{P}_i$ has an open graph, thus Φ_i^n is lower semi continuous. We now apply, Gale - Mas-Colell fixed point theorem to the correspondences $(\Phi_i^n)_{i=0}^I$, we obtain the following result.

<u>CLAIM</u> 5.2 *For any integer $n \geq 1$, there exists $(p^n, x^n, z^n) \in \mathbb{P}^n \times \tilde{X} \times \mathbb{Z}^{nI}$ such that:*

$$\forall i \in I, B_i(p^n, x^n, z^n) \cap \tilde{P}_i(x^n) \times \mathbb{Z}^n = \emptyset; \tag{4}$$

$$p^n \cdot \sum_{i \in I}(x_i^n - e_i) \geq p \cdot \sum_{i\in I}(x_i^n - e_i), \ \forall p \in \mathbb{P}^n; \tag{5}$$

$$\forall i \in I, \ p^n \,\square\, (x_i^n - e_i) \leq W\, z_i^n + \alpha(p^n) + \varphi(p^n, x^n, z^n). \tag{6}$$

Set $d^n = \alpha(p^n) + \varphi(p^n, x^n, z^n)$. We can extract a subsequence such that p^n converges to p, x^n converges to x, for all $s \in \mathbb{S}$, d_s^n converges in $\overline{\mathbb{R}}$ to d_s and for all $i \in I$, all $s \in \mathbb{S}$, $W_s z_i^n$ converges in $\overline{\mathbb{R}}$.

<u>CLAIM</u> 5.3 *For all $s \in \mathbb{S}$, such that for some $i \in I$ $W_s z_i^n$ does not converge in $\mathbb{R}$, we have $d_s = +\infty$.*

<u>*Proof.*</u> Let $i \in I$ and $s \in \mathbb{S}$ such that $W_s z_i^n$ converges to $-\infty$. Since prices and allocations remain in a compact subset, we must have that d_s^n converges to $+\infty$. Otherwise consumer i violates his budget constraint (6) in s for n large enough. Let $i \in I$ and $s \in \mathbb{S}$ such that $W_s z_i^n$ converges to $+\infty$. Then either there exist $i' \in I$ such that $W_s z_{i'}^n$ converges to $-\infty$ and therefore d_s^n converges to $+\infty$. Otherwise $\varphi_{i's}(p^n, x^n, z^n)$ converges to $+\infty$ and we have also that d_s^n converges to $+\infty$. $\square$

<u>CLAIM</u> 5.4 *There exists $k > 0$, $z^1, \ldots, z^k \in (\mathbb{R}^J)^I$, and sequences $(\varepsilon_r^n)_n$ of $\mathbb{R}_+$, $r = 1, \ldots, k$ such that for some subsequence.*

$$z^n = \sum_{r=1}^k \varepsilon_r^n z^r$$

with $\left(\dfrac{\varepsilon_{r+1}^n}{\varepsilon_r^n}\right)_n$ *converging to zero for all* $r \in \{1, \ldots, k-1\}$.

<u>*Proof.*</u> Set $z^n = z_1^n$. For $r \in \{1, \ldots, JI\}$, if for a subsequence, $z_r^n = 0$, then $z^r = \ldots = z^{JI} = 0$. Otherwise take a subsequence such that $\|z_r^n\|$ and $z_r^n/\|z_r^n\|$ converge in $\overline{\mathbb{R}}$. Let z^r be the limit of $z_r^n/\|z_r^n\|$. Let

$$\mathcal{H}^r = \{x \in R^{JI} \mid z^r \cdot x = 0\}.$$

For $r \in \{2, \ldots, JI\}$, let

$$z_r^n = \mathrm{proj}_{\mathcal{H}^{r-1}}(z_{r-1}^n).$$

Let $k = \min\{1 \in \{1, \ldots, JI\} \mid z^{r+1} = \ldots = z^{JI} = 0\}$. So k is at most equal to JI. Note that for all $r \in \{1, \ldots, k\}$, $\| z_{r+1}^n \| = \|z_r^n\| o(\| p_r^n \|)$ [10].

We can thus decompose the sequence z^n in the following way:

$$z^n = \sum_{r=1}^{k} (\| z_r^n \| - \| z_{r+1}^n \|) z^r = \sum_{r=1}^{k} \varepsilon_r^n z^r,$$

with $\varepsilon_{r+1}^n = \varepsilon_r^n o(\varepsilon_r^n)$ for $r \in \{1, \ldots, k-1\}$. □

Set $\kappa = k+1$ if for all $r \in \{1, \ldots, k\}$, ε_r^n converges to $+\infty$, otherwise let κ be the smallest $r \in \{1, \ldots, k\}$ such that ε_r^n does not converge to $+\infty$. For all $i \in I$, set $\zeta_i^n = 0$ if $\kappa = k+1$. Otherwise set $\zeta_i^n = \sum_{r=\kappa}^{k} \varepsilon_r^n z_i^r$. Set $\zeta_i = \lim \zeta_i^n$.

<u>Claim</u> 5.5 *For all large enough* n*, for all* $i \in I$*,* $(x_i^n, \zeta_i^n) \in B_i(p^n, x^n, z^n)$*.*

<u>*Proof.*</u> This is simply due to the fact that for all $s \in \mathbb{S}$ such that for some $r < \kappa$, $W_s z_i^r \neq 0$ implies that d_s^n converges to $+\infty$ and that the allocations and prices remain in a compact set. □

<u>Claim</u> 5.6 *If for a subsequence of* n*,* $\sum_{i \in I} x_i^n = e$*, then for all large* n*,* $(x^n, \bar{z}^n, p^n, q, \bar{d}^n)$ *where for* $i = 1, \ldots, I$

$$\bar{z}_i^n = \zeta_i^n - \frac{1}{I} \sum_{\iota \in I} \zeta_\iota^n$$
$$\bar{d}^n = d^n + \frac{1}{I} W \sum_{\iota \in I} \zeta_\iota^n$$

is a financial equilibrium with slack.

[10] We denote by $o : \mathbb{R} \to \mathbb{R}$ a function which is continuous in 0 with $o(0) = 0$.

Proof. Note first that for all $s \in \mathbb{S}$, if d^n_s converges to $+\infty$, then $\bar{d}^n_s$ converges also to $+\infty$. For all other $s \in \mathbb{S}$, $W_s \sum_{i \in I} z^n_i = W_s \sum_{i \in I} \zeta^n_i$. Thus, since

$$0 = p^n \,\square \sum_{i \in I}(x^n_i - e_i) \leq \sum_{i \in I} W\, z^n_i + I\alpha(p^n) + I\varphi(p^n, x^n, z^n), \tag{7}$$

We have that for all large n, $0 \leq \bar{d}^n$. We need to check that

$$B^F_i(p^n, q, d^n_i) = B^F_i(p^n, q, \bar{d}^n).$$

Let $y \in B^F_i(p^n, q, d^n)$, then there exists $z' \in \mathbb{R}^J$ such that

$$p^n \,\square\, (y - e_i) \leq W\, z' + d^n,$$

so $p^n \,\square\, (y - e_i) \leq W\, \left(z' - \frac{1}{I}\sum_{i \in I} \zeta^n_i\right) + \bar{d}^n$ and therefore $y \in B^F_i(p^n, q, \bar{d}^n)$.

Conversely, let $y \in B^F_i(p^n, q, \bar{d}^n)$, then there exists $z' \in \mathbb{R}^J$ such that

$$p^n \,\square\, (y - e_i) \leq W\, z' + \bar{d}^n = W\, \left(z' + \frac{1}{I}\sum_{i \in I} \zeta^n_i\right) + d^n$$

and therefore $y \in B^F_i(p^n, q, d^n)$.

Note that $\hat{P}_i(\xi) \neq \emptyset$ implies ξ is in the closure of $\hat{P}_i(\xi)$. This property holds also for $\tilde{P}_i$ as long as $\|\xi\| < r$. Moreover both preference relations are convex valued. Thus, since for all $i \in I$, $\|x^n_i\| < r$ and for some $\bar{n}$ and all $n \geq \bar{n}$, $\|\zeta^n_i\| < \bar{n}$, standard arguments show that for all large n,

$$(\tilde{P}_i(x^n) \times \mathbb{Z}^n) \cap B_i(p^n, x^n, z^n) = \emptyset \Rightarrow \hat{P}_i(x^n) \cap B^F_i(p^n, q, d^n) \neq \emptyset.$$

□

We now treat the case where for all $n \geq 1$, $\sum_{i \in I}(x^n_i - e_i) \neq 0$.

Claim 5.7 *$\|p\| = 1$, $\sum_{i \in I} \zeta_i = 0$, $\sum_{i \in I} x_i = e$ and for all $i \in I$,*

$$p \,\square\, (x_i - e_i) \leq W\, z_i + d. \tag{8}$$

Proof. Since for all $n \geq 1$, $\sum_{i \in I} x^n_i \neq e$, we have from (5), $\|p^n\| = 1 - \frac{1}{n}$ and thus at the limit one obtains $\|p\| = 1$ and $\alpha(p) = 0$. Let $s \in \mathbb{S}$, then either there exists a subsequence and some $i_0 \in I$ such that for all n, $\varphi_{i_0 s}(p^n, x^n, z^n) > 0$. Without loss of generality we may choose i_0 and the subsequence such that for all n, $\varphi_{i_0 s}(p^n, x^n, z^n) = \varphi_s(p^n, x^n, z^n)$. Then, we obtain from the budget constraint of consumer i_0 that

$$p_s \cdot (x^n_{i_0 s} - e_{i_0 s}) \leq \left(W\, z^n_{i_0}\right)_s + \sum_{i \in I \setminus \{i_0\}} [(W\, z^n_i)_s - p_s \cdot (x^n_{is} - e_{is})] + 1/n$$

and thus

$$\sum_{i\in I} p_s^n \cdot (x_{is}^n - e_{is}) \leq \left(W \sum_{i\in I} z_i^n \right)_s + 1/n.$$

Otherwise, for a subsequence for all $i \in I$, $\varphi_{is}(p^n, x^n, z^n) \leq 0$. Then,

$$p_s^n \cdot \sum_{i\in I} (x_{is}^n - e_{is}) \leq \left(W \sum_{i\in I} z_i^n \right)_s + 1/n.$$

Since by (5)

$$p^n = \lambda^n \sum_{i\in I} (x_i^n - e_i)$$

with $\lambda^n > 0$, for all n for all $s \in \mathbb{S}$,

$$0 \leq p_s^n \cdot \sum_{i\in I} (x_{is}^n - e_{is}).$$

Thus,

$$-\frac{1}{n}(1, \ldots, 1) \leq W \sum_{i\in I} z_i^n.$$

If $\sum_{i\in I} z_i^n$ does not converge to 0, then, since W is one-to-one, $\|W \sum_{i\in I} z_i^n\|$ does not include 0 in the set of its accumulation points and for a subsequence $\dfrac{\sum_{i\in I} z_i^n}{\left\| W \sum_{i\in I} z_i^n \right\|}$ converges to some $\bar{z}$ with $\|W\bar{z}\| = 1$. Therefore $0 \leq W\bar{z}$ and this contradicts the non-arbitrage condition. Hence $\sum_{i\in I} z_i^n$ converges to 0 and therefore $\sum_{i\in I} \zeta_i = 0$. Thus, we have that for all $s \in \mathbb{S}$, $p_s \,\square\, \sum_{i\in I}(x_{is} - e_{is}) = 0$, and by (5), $p^n = \lambda^n \sum_{i\in I}(x_i^n - e_i)$ with $\lambda^n > 0$ for all n, one obtains, $\sum_{i\in I}(x_i - e_i) = 0$. $\square$

In the following, we will distinguish different situations and consider the following subsets of states of the nature $\mathbb{S}$.

$$\mathbb{S}^0 = \{ s \in \mathbb{S} \,|\, p_s = 0 \};$$

$$\mathbb{S}^+ = \{ s \in \mathbb{S} \,|\, p_s \neq 0 \};$$

$$\Pi^+(n) = \left\{ s \in \mathbb{S} \,\middle|\, p_s^n \cdot \sum_{i\in I} (x_{is}^n - e_{is}) = \left(W \sum_{i\in I} z_i^n + 1/n \right)_s \right\};$$

$$\Pi^-(n) = \left\{ s \in \mathbb{S} \,\middle|\, p_s^n \cdot \sum_{i\in I} (x_{is}^n - e_{is}) < \left(W \sum_{i\in I} z_i^n + 1/n \right)_s \right\}.$$

Since $\mathbb{S}$ is finite, there exist two sets Π^+ and Π^- and a subsequence such that for all n of this subsequence $\Pi^+(n) = \Pi^+$ and $\Pi^-(n) = \Pi^-$. From now on we restrict ourselves to this subsequence.

<u>CLAIM</u> 5.8 $\quad \mathbb{S} = \Pi^- \cup \Pi^+$.

<u>Proof.</u> If for some $s \in \mathbb{S}$ and some $n \in \mathbb{N}$,

$$p_s^n \cdot \sum_{i \in I}(x_{is}^n - e_{is}) > \left(W \sum_{i \in I} z_i^n + 1/n \right)_s, \tag{9}$$

then for every $i \in \{1, \dots, I\}$,

$$p_s^n \cdot (x_{is}^n - e_{is}) > (W\, z_i^n)_s + \alpha_s(p^n) + \varphi_{is}(p^n, x^n, z^n)$$

and thus, by the individual budget constraints, for all $i \in I$, $\varphi_{is}(p^n, x^n, z^n) < 0$. Thus, by summing up the individual budget constraints for state s one obtains

$$p_s^n \cdot \sum_{i \in I}(x_{is}^n - e_{is}) \leq \left(W \sum_{i \in I} z_i^n \right)_s + I\alpha_s(p^n)$$

which yields to a contradiction with (9). $\square$

<u>CLAIM</u> 5.9 $\quad$ *If $s \in \Pi^-$, then, for all $i \in I$ and all n,*

$$p_s^n \cdot (x_{is}^n - e_{is}) < (W\, z_i^n)_s + \alpha_s(p^n) + \varphi_s(p^n, x^n, z^n).$$

<u>Proof.</u> Since $s \in \Pi^-$, one has for all n :

$$p_s^n \cdot \sum_{i \in I}(x_{is}^n - e_{is}) < \left(W \sum_{i \in I} z_i^n + I\alpha_s(p^n) \right)_s.$$

Thus, $\forall i \in I$,

$$\begin{array}{ll} & p_s^n \cdot (x_{is}^n - e_{is}) \\ < & (W\, z_i^n)_s + \alpha_s(p^n) + \varphi_{is}(p^n, x^n, z^n) \\ \leq & (W\, z_i^n)_s + \alpha_s(p^n) + \varphi_s(p^n, x^n, z^n). \end{array}$$

$\square$

<u>CLAIM</u> 5.10 $\quad$ *There exists $\hat{n}$, such that : $\forall s \in \mathbb{S}^0 \bigcap \Pi^+$ and $\forall n \geq \hat{n}$:*

$$\left(W \sum_{i \in I} z_i^n \right)_s < 0.$$

Proof. Remark first that $\mathbb{S}^+ \neq \emptyset$ since $\|p\| = 1$. Without any loss of generality, we can assume that for all $s \in \mathbb{S}^+$, $p_s^n \neq 0$, $\forall n \in \mathbb{N}$. Consider

$$G_s^n = p_s^n \cdot \sum_{i \in I} (x_{is}^n - e_{is}) \qquad \text{and} \qquad H_s^n = \left(W \sum_{i \in I} z_i^n \right)_s .$$

Remark that from statement (5), there exists a real number $\lambda^n > 0$ such that

$$\sum_{i \in I} (x_i^n - e_i) = \lambda^n p^n.$$

Thus for all $(s^0, s^+) \in \left(\mathbb{S}^0 \bigcap \Pi^+\right) \times \mathbb{S}^+$, one has :

$$\lim_{n \to +\infty} \frac{G_{s^0}^n}{G_{s^+}^n} = \lim_{n \to +\infty} \frac{\|p_{s^0}^n\|^2}{\|p_{s^+}^n\|^2} = 0$$

One has also from statement (5), that for all $s \in \mathbb{S}$ and for all n :

$$0 \leq G_s^n \leq H_s^n + \frac{1}{n}$$

and we have the equality if and only if $s \in \Pi^+$. Since $s^0 \in \Pi^+$, one has,

$$0 \leq \lim_{n \to +\infty} \frac{H_{s^0}^n + \frac{1}{n}}{H_{s^+}^n + \frac{1}{n}} \leq \lim_{n \to +\infty} \frac{G_{s^0}^n}{G_{s^+}^n} = 0 \tag{10}$$

Thus if for some subsequence $H^n = 0$ (in particular if there are no assets), then $\Pi^+ \cap \mathbb{S}^0 = \emptyset$ and so the claim is trivially satisfied.

Otherwise, suppose there exists $s^0 \in \mathbb{S}^0 \bigcap \Pi^+$ such that for some subsequence, $H_{s^0}^n \geq 0$. For the rest of the proof of the claim we work with this subsequence. Let $s^+ \in \mathbb{S}^+$, then for all n,

$$\frac{H_{s^0}^n + \frac{1}{n}}{H_{s^+}^n + \frac{1}{n}} \geq \frac{\frac{1}{n}}{H_{s^+}^n + \frac{1}{n}} = \frac{1}{nH_{s^+}^n + 1} \geq 0.$$

Thus from (10), $\lim_{n \to +\infty} nH_{s^+}^n = +\infty$ and $H_{s^+}^n > 0$ for n large enough. Thus, one has for all n large enough

$$\frac{H_{s^0}^n + \frac{1}{n}}{H_{s^+}^n + \frac{1}{n}} \geq \frac{H_{s^0}^n + \frac{1}{n}}{2H_{s^+}^n} \geq \frac{H_{s^0}^n}{2H_{s^+}^n} \geq 0 \tag{11}$$

and therefore

$$\lim_{n \to +\infty} \frac{H_{s^0}^n}{H_{s^+}^n} = 0 \quad \Longrightarrow \quad \lim_{n \to +\infty} \frac{H_{s^0}^n}{\|H^n\|} = 0. \tag{12}$$

Let $\bar{z}^n = \dfrac{\sum_{i \in I} z_i^n}{\|H^n\|}$. Remark that for all n, $\|W\,\bar{z}^n\| = 1$, thus $(W\,\bar{z}^n)$ has a subsequence which is converging to a vector of $\mathbb{R}^{S+1}$ which is of the form $W\,\bar{z}$. It is clear that $\bar{z} = \lim_{n\to+\infty} \bar{z}^n$ and that $\|W\,\bar{z}\| = 1$. The previous analysis implies the following :

$$\begin{cases} (W\,\bar{z})_s & \leq \;\; 0 \qquad \text{for all } s \in \mathbb{S}^0 \bigcap \Pi^+; \\ (W\,\bar{z})_s & \geq \;\; 0 \qquad \text{for all } s \in \mathbb{S}^+; \end{cases}$$

Since $W\,\bar{z} \neq 0$, from non-arbitrage, one has the existence of $s^* \in \mathbb{S}^0$ such that $(W\,\bar{z})_{s^*} < 0$. Thus, for all $s^+ \in \mathbb{S}^+$, $\lim_{n\to+\infty} \dfrac{H^n_{s^*}}{H^n_{s^+}} \in [-\infty, 0[$ and since $\lim_{n\to+\infty} nH^n_{s^+} = +\infty$, one deduces that $\lim_{n\to+\infty} nH^n_{s^*} = -\infty$ and therefore for n large enough

$$0 \leq p^n_{s^*} \cdot \sum_{i \in I} (x^n_{is^*} - e_{is^*}) \leq H^n_{s^*} + \frac{1}{n} < 0.$$

The first inequality is implied by (5). Thus we have a contradiction. □

<u>Claim</u> 5.11 *For $i \in I$, $B_i^F(p,q,d) \bigcap P_i(x) = \emptyset$.*

<u>*Proof.*</u> We proceed by contraposition and assume there exists $(y, \zeta') \in X_i \times \mathbb{R}^J$ with

$$y \in P_i(x) \quad \text{and} \quad p \,\Box\, (y - e_i) \leq W\zeta' + d.$$

We claim that we may assume without any loss of generality that for some $\varepsilon > 0$,

$$\forall s \in \mathbb{S}^+ \cup \{s \in \mathbb{S} \mid d_s > 0\}, \quad p_s \cdot (y_s - e_{is}) \leq \left(W\zeta'\right)_s + d_s - 2\varepsilon. \tag{13}$$

To prove assertion (13), set for all $s \in \mathbb{S}$, $X_{is}(y) = X_i \cap \left(\{y_{-s}\} \times \mathbb{R}^\ell\right)$ and proceed with the following steps.

- For all $s \in \mathbb{S}^+$ such that $p_s \cdot y_s > \min_{x \in X_{is}(y)} p_s \cdot x_s$, there exists $t_s \in \mathbb{R}^\ell$ with $p_s \cdot t_s = 1$ such that $y - \varepsilon_s(0, \dots, 0, t_s, 0, \dots, 0) \in X_i$ for all $\varepsilon_s > 0$ small enough. For all other $s \in \mathbb{S}$ set $t_s = 0$. By the convexity of consumption sets, one can choose $\varepsilon > 0$, such that, $y(\varepsilon) = y - \varepsilon(t_0, \dots, t_S) \in X_i$. Furthermore for ε can be chosen small enough one has $y(\varepsilon) \in \hat{P}_i(x)$.

- For all $s \in \mathbb{S}^+$ such that $p_s \cdot y_s = \min_{x \in X_{is}(y)} p_s \cdot x_s$, one has by Assumption S $p_s \cdot (y_s(\varepsilon) - e_{is}) = p_s \cdot (y_s - e_{is}) < 0$. Thus for $\eta \in]0,1[$, small enough, $p_s \cdot (y_s(\varepsilon) - e_{is}) < (1-\eta)\,(W\zeta')_s + d_s$.

- Taking $\eta > 0$ small enough, we have

$$\forall s \in \mathbb{S}^+ \cup \{s \in \mathbb{S} \mid d_{is} > 0\}, \quad p_s \cdot (y_s(\varepsilon) - e_{is}) \leq (1-\eta)\left(W\zeta'\right)_s + d_s - 2\varepsilon.$$

- Note that for $\forall s \in \mathbb{S}^0 \cap \{s \in \mathbb{S} \mid d_s = 0\}$, we have $(W\zeta')_s = 0$ and thus

$$p_s \cdot (y_s(\varepsilon) - e_{is}) \leq (1-\eta)\left(W\zeta'\right)_s \leq \left(W\zeta'\right)_s.$$

This ends the proof of assertion (13) by replacing y by $y(\varepsilon)$ and ζ' by $(1-\eta)\zeta'$.

Now, we may extract a subsequence such that for all n, $y \in \hat{P}_i(x^n)$ (since the graph of $\hat{P}_i$ is open) and for some fixed $\bar{s} \in S^0$, $\|p^n_{\bar{s}}\| \geq \|p^n_s\|$, for all n and $s \in \mathbb{S}^0$.

We show that some n large enough, there exists $\lambda \in]0,1]$ and $\bar{\zeta} \in \mathbb{Z}^n$, such that,

$$\begin{cases} (\lambda y + (1-\lambda)x^n_i, \bar{\zeta}) \in B_i(p^n, x^n, z^n) \\ \\ \lambda y + (1-\lambda)x^n_i \in \tilde{P}_i(x^n). \end{cases} \tag{14}$$

First remark that by Claim 5.10 there exists $\eta > 0$ such that for all $s \in \mathbb{S}^0 \cap \Pi^+$:

$$p^n_s \cdot (y_s - e_{is}) < \left(W(\zeta' + \eta^n\delta)\right)_s + d^n_s$$

where $\delta = -\sum_{i\in I} z^{\bar{n}}_i$ for some fixed $\bar{n} \geq \hat{n}$ and $\eta^n = \|p^n_{\bar{s}}\|\eta$. From this and from assertion (13), one has for all $\lambda \in]0,1[$ small enough

$$\forall s \in \mathbb{S}^+ \cup \{s \in \mathbb{S} \mid d_s > 0\} \cup (\mathbb{S}^0 \cap \Pi^+) \text{ and all large } n, \text{ one has :}$$

$$p^n_s \cdot (\lambda y_s + (1-\lambda)x^n_{is} - e_{is}) \leq \left(W\left(\lambda(\zeta' + \eta^n\delta) + (1-\lambda)\zeta^n_i\right)\right)_s + d^n_s. \tag{15}$$

Furthermore, by claim (5.9), one can choose $\lambda > 0$ such that inequality (15) still holds true for $s \in \mathbb{S}^0 \cap \Pi^-$ say for some n large enough.

Since $(x^n_i, \zeta^n_i) \in \mathrm{int}\tilde{X}_i \times \mathrm{int}\mathbb{Z}^n$, $(\lambda y + (1-\lambda)x^n_i) \in \tilde{X}_i$ and $\bar{\zeta} = (\lambda(\zeta' + \eta^n\delta) + (1-\lambda)\zeta^n_i) \in \mathbb{Z}^n$. Thus, $(\lambda y + (1-\lambda)x^n_i, \bar{\zeta}) \in B_i(p^n, x^n, z^n)$ and $\lambda y + (1-\lambda)x^n_i \in \tilde{P}_i(x^n)$ leading to a contradiction. □

6 Proof of Theorem 4.1

We start by fixing an arbitrage free price $q \in \mathbb{R}^J$ and we introduce an auxiliary agent b with $e_b = 0$, $X_b = \mathbb{R}^{\ell(S+1)}_+$ and whose preferences are representable by the function $u_b : X_b \to \mathbb{R}$ defined by $u_b(x) = \|x\|_1$. We extend the measure space to of agents M', by adding an atom of weight $\mu(b) = 1$ composed of the additional consumer ($M' = M \cup \{b\}$). Let $n > \operatorname{ess\,sup}_{a\in M'} \|\hat{x}(a)\|$, and let :

$$X^n(a) = X(a) \cap B(0,n), \forall a \in M';$$

$$\mathbb{Z}^n = \{z \in \mathbb{R}^J \mid \|z\| \leq n\};$$
$$X^n = \int_M X^n(a) d\mu(a);$$

For all $(a,p) \in M \times \mathbb{R}^{\ell(S+1)}$, Let :

$$B^n(a,p) = \left\{(x,z) \in X^n(a) \times \mathbb{Z}^n \mid \; p \,\square\, (x - e(a)) \leq W\, z + \frac{1}{n}\,(1,\ldots,1)\right\}$$
$$d^n(a,p) = \{(x,z) \in B^n(a,p) \mid [P_a(x) \times \mathbb{Z}^n] \cap B^n(a,p) = \emptyset\}$$
$$\delta^n_a(p) = \int_M d^n(a,p) d\mu(a) - e$$

From Assumption S', $p \,\square\, e(a) + \frac{1}{n} \gg \min p \,\square\, X^n(a)$, for all $p \in \mathbb{P}$ and for a.e. a in M. For $(p,x,z) \in \mathbb{P} \times X^n \times \mathbb{Z}^n$ let

$$\varphi(p,x,z) = \left(\max\left\{0, \frac{1}{n} + W_s z + p_s \cdot (e_s - x_s)\right\}\right)_{s \in \mathbb{S}},$$
$$B^n(b,p,x,z) = \{y \in X^n(b) \,|\, p \,\square\, y \leq \varphi(p,x,z)\};$$
$$\delta^n_b(p,x,z) = \left\{y \in B^n(b,p,x,z) \,\middle|\, \|y_s\|_1 < \|y'_s\|_1 \Rightarrow \varphi_s(p,x,z) \leq p_s \cdot y'_s,\; \forall s \in \mathbb{S}\right\}.$$

For $(p,x,\xi,z) \in \mathbb{P} \times X^n \times \mathbb{X}^n(b) \times \mathbb{Z}^n$ let

$$\delta^n_0(p,x,\xi,z) = \operatorname{argmax}\left\{p' \cdot (x + \xi - e) \mid p' \in \mathbb{P}\right\}.$$

Let $\delta^n = (\delta^n_0, \delta^n_a, \delta^n_b)$ defined from $\mathbb{P} \times X^n \times \mathbb{Z}^n \times X^n(b)$ to itself.

Standard arguments show that the correspondences δ^n are upper semi-continuous with non-empty, convex, compact values. Thus for all n there exists a fixed point $(p^n, x^n, \xi^n, z^n) \in \mathbb{P} \times X^n \times X^n(b) \times \mathbb{Z}^n$ of the correspondence δ^n. Thus for all n large enough, there exists $(p^n, x^n(.), \xi^n z^n(.)) \in \mathbb{P}^n \times S^1_X \times X^n(b) \times L^1(M, \mathbb{R}^J)$ such that the following holds (setting $x^n(b) = \xi^n$) :

$$\text{For a.e } a \in M, \quad \begin{cases} p^n \,\square\, (x^n(a) - e(a)) \leq W z^n(a) + \frac{1}{n}(1,\ldots,1) \\ [P(a, x^n(a)) \times \mathbb{Z}^n] \cap B^n(a,p^n) = \emptyset \end{cases} \tag{16}$$

$$\begin{array}{ll} & p^n \,\square\, (x^n(b) - e(b)) \leq \varphi(p^n, x^n, z^n) \\ \forall s \in \mathbb{S}, & \|y_s\|_1 > \|x^n_s(b)\|_1 \quad \Rightarrow \quad p^n_s \cdot y_s \geq \varphi_s(p^n, x^n, z^n) \end{array} \tag{17}$$

$$\forall p \in \mathbb{P}, \quad (p^n - p) \cdot \int_{M'} x^n(a) d\mu(a) \geq 0 \tag{18}$$

<u>Claim</u> 6.1 *The sequences* $\left(\int_{M'}(x^n(a) - e(a)d\mu(a)\right)_n$ *and* $\left(\int_M z^n(a)d\mu(a)\right)_n$ *converge to* 0 *and there exists* n_0*, such that :* $\forall n \geq n_0$,

$$p^n \,\square\, \int_{M'}(x^n(a) - e(a)d\mu(a) = W \int_M z^n(a)d\mu(a) + \frac{1}{n}(1,\ldots,1).$$

Proof. From (16), one has

$$p^n \square \left(\int_M (x^n(a) - e(a)) d\mu(a) \right) \le W \int_M z^n(a) d\mu(a) + \frac{1}{n}(1, \dots, 1)$$

which implies in particular that $\varphi(p^n, x^n, z^n) =$

$$W \int_M z^n(a) d\mu(a) + \frac{1}{n}(1, \dots, 1) - p^n \square \left(\int_M (x^n(a) - e(a)) d\mu(a) \right) \tag{19}$$

Thus, from (17) and (18), one in addition obtains that $\forall n$:

$$0 \le p^n \square \left(\int_{M'} x^n(a) - e(a) \right) \le W \int_M z^n(a) d(\mu(a) + \frac{1}{n}(1, \dots, 1) \tag{20}$$

which implies in particular that $-\frac{1}{n}(1, \dots, 1) \le W \int_M z^n(a) d(\mu(a)$. Since W is arbitrage free, one can select a subsequence such that $\int_M z^n(a) d\mu(a)$ converges to 0 and hence, also, $p^n \square \int_{M'} (x^n(a) - e(a)) d\mu(a)$ converges to 0 which from (18) implies that $\int_{M'} (x^n(a) - e(a)) d\mu(a)$ converges to 0.

Since $\int_M x^n(a) d\mu(a)$ and ξ^n are bounded from below, and since the allocation of commodities is feasible at the limit, one deduces that ξ^n converges to some ξ^* for some appropriate subsequence and there exists n_0 such that for all $n \ge n_0$, $\|\xi^n\|_1 < n_0 < n$ meaning that, one has the equality in each constraint of agent b for all $n > n_0$:

$$p^n \square \int_{M'} (x^n(a) - e(a) d(\mu(a)) = W \int_M z^n(a) d(\mu(a) + \frac{1}{n}(1, \dots, 1).$$

□

Remark that since (p^n) remains in a compact set, we assume without any loss of generality that it converges to p^*. Moreover, the preferences of agent b ensure that for all $n > n_0$, $p^n \in \mathbb{R}_+^{\ell(S+1)}$ implying that $p^* \in \mathbb{R}_+^{\ell(S+1)}$. Furthermore, from (18), either for a subsequence we have $\int_{M'} x^n(a) d\mu(a) = e$ or $\|p^n\| = 1$. If $\int_{M'} x^n(a) d\mu(a) = e$ holds for a subsequence, then by the previous claim for all $n \ge n_0$,

$$0 = W \int_M z^n(a) d\mu(a) + \left(\frac{1}{n}, \dots, \frac{1}{n} \right).$$

But, this contradicts the non-arbitrage condition and thus Thus

$$\begin{array}{ll} \forall n > n_0, & \int_{M'} x^n(a) d\mu(a) \ne e \\ \text{and} & \|p^*\| = 1 \end{array}$$

We now apply Fatou's lemma in several dimensions to the sequence $(x^n(.), Wz^n(.))$ of integrable functions from M to $\mathbb{R}^{\ell(S+1)} \times \mathbb{R}^{S+1}$. From the above analysis, this sequence is bounded from below by an integrable function and its integral is convergent.

Thus : there exist $x^*(.) : M \to \mathbb{R}^{\ell(S+1)}$ and $t^*(.) : M \to \mathbb{R}^{S+1}$ integrable functions such that

$$\int_M x^*(a)d\mu(a) \leq e; \tag{21}$$

$$\int_M t^*(a)d\mu(a) \leq 0; \tag{22}$$

$$\text{for a.e. } a \in M, \quad (x^*(a), t^*(a)) \text{ is adherent to } (x^n(a), Wz^n(a)) \tag{23}$$

From (23), one has that for a.e., $a \in M$, the existence of $z^*(a) \in \mathbb{R}^J$ such that $t^*(a) = Wz^*(a)$. Since W is injective, $z^*(.)$ is integrable and from (22) one has $W\left(\int_M z^*(a)d\mu(a)\right) \leq 0$. But from the non-arbitrage condition, one has the equality meaning in particular that $\int_M z^*(a)d\mu(a) = 0$.

Let $\mathbb{S}^0 = \{s \in \mathbb{S} \,|\, p^*_s = 0\}$ and $\mathbb{S}^+ = \{s \in \mathbb{S} \,|\, p^*_s \neq 0\}$.

<u>Claim</u> 6.2 *There exists $n_1 \in \mathbb{N}$, such that : $\forall s \in \mathbb{S}^0$ and $\forall n \geq n_1$:*

$$\left(W \int_M z^n(a)d\mu(a)\right)_s < 0.$$

For a proof, we adapt the arguments from claim (5.10).

<u>Claim</u> 6.3 *For a.e. $a \in M$, $[P_a(x^*(a)) \times \mathbb{R}^J] \cap B(a, p^*) = \emptyset$.*

<u>*Proof.*</u> We proceed by contraposition. Let $\mathcal{N}$ be the set of agents in M' who do not consume a maximal element at some step n or for whom $(x^*(a), Wz^*(a))$ is not adherent to $(x^n(a), Wz^n(a))$. Since $\mathcal{N}$ is a countable union of negligible sets, it is negligible. Suppose that for some $a \in M \setminus \mathcal{N}$, $(y, \zeta) \in [P_a(x^*(a)) \times \mathbb{R}^J] \cap B(a, p^*)$. Since, by assumption C$'$, the graph of P_a is open, we can proceed as in claim (5.5) such that for some fixed $\varepsilon > 0$:

$$\forall s \in \mathbb{S}^+, \quad p^*_s \cdot (y_s - e_s(a)) \leq (W\zeta)_s - 2\varepsilon$$

We can also assume without any loss of generality that for all large n, $y \in P_a(x^n(a))$, $\zeta \in \mathbb{Z}^n$ and for some fixed $\bar{s} \in \mathbb{S}^0$, $\|p^n_{\bar{s}}\| \geq \|p^n_s\|$, for all n and $s \in \mathbb{S}^0$.

First remark that there exists $\eta \in \mathbb{R}_+$ such that for all $s \in \mathbb{S}^0$:

$$p^n_s \cdot (y_s - e_s(a)) < (W(\zeta + \eta^n\delta))_s$$

where $\delta = -\int_M z^{\bar{n}}(a)d\mu(a)$ for some $\bar{n} \geq \hat{n}$ and $\eta^n = \|p^n_{\bar{s}}\|\eta$. And for n large enough, one has that for all $s \in S^+$,

$$p^n_s \cdot (y_s - e_s(a)) < (W(\zeta + \eta^n\delta))_s \,.$$

Thus $(y, \zeta + \eta^n\delta) \in B^n(a, p^n)$ for n large enough. There is a contradiction since for n large enough, $y \in P_a(x^n(a))$. □

References

[1] R. Aumann and J. Drèze, *Values of markets with satiation or fixed prices*, Econometrica, (1986), 54, pp. 1271–1318.

[2] D. Cass, *Competitive equilibrium with incomplete financial markets*, CARESS WP 84-09, (1984).

[3] D. Cass, *On the 'Number' of Equilibrium Allocations with Incomplete Financial markets*, CARESS WP 85-16, (1985), University of Pennsylvania.

[4] J. Drèze and H. Müller, *Optimality properties of rationing schemes*, Journal of Economic Theory, (1980), 23, pp. 150–159.

[5] D. Duffie and W. Shafer, *Equilibrium in incomplete markets ii: Genereic existence in stochastic economies*, Journal of Mathematical Economics, (1986), 15, pp. 199–216.

[6] D. Gale and A. Mas-Colell, *An Equilibrium Existence Theorem for a General Mol without Ordered Preferences*, Journal of Mathematical Economics, (1975), 2, pp. 9–15.

[7] D. Gale and A. Mas-Colell, *Corrections to an Equilibrium Existence Theorem for a General Model without Ordered Preferences*, Journal of Mathematical Economics, 1979, 6, pp. 297–298.

[8] J.-D. Geanakoplos and H.-M. Polemarchakis, *Existence, regularity, and constrained suboptimality of competitive allocations when the asset market is incomplete*, Heller,-Walter-P.,-ed and Starr,-Ross-M.,-ed and Starrett,-D.-A.,-ed. Essays in honor of Kenneth J. Arrow. Volume 3. Uncertainty, information, and communication. Cambridge; New York and Sydney: Cambridge University Press, (1986), pp. 65–95.

[9] P. Gottardi and T. Hens, *The survival assumption and existence of competitive equilibria when asset markets are incomplete*, Journal of Economic Theory; 71(2), November, (1996), pp. 313–23.

[10] A. Kahji, *How to discard non-satiation and free disposal with paper money*, Journal of Mathematical Economics, (1996), 25, pp. 75–84.

[11] V. Makarov, *Some results on general assumptions about the existence of economic equilibrium*, Journal of Mathematical Economics, (1981), 8, pp. 87–100.

[12] A. Mas-Colell, *Equilibrium theory with possibly satiated preferences*, in Equilibrium and dynamics: Essays in honour of David Gale, edited by M. Majumdar, (1992), pp. Macmillan, London, pp. 201–213.

[13] H. Polemarchakis and P. Siconolfi, *Competitive equilibria without free disposal or nonsatiation*, Journal of Mathematical Economics, (1993), 22, pp. 85–99.

[14] H. Polemarchakis and P. Siconolfi, *Generic existence of competitive equilibria when the asset market is incomplete: a symmetric argument*, Economic Theory, (1995), 6, pp. 495–510.

[15] J. Werner, *Equilibrium in economies with incomplete financial markets*, Journal of Economic Theory, (1985), 36, pp. 110–119.