ANNALES D’ÉCONOMIE ET DE STATISTIQUE. – N° 61 – 2001

On Irreducible Economies

Michael FLORIG *

ABSTRACT. – Without a survival assumption, a Walras equilibrium may fail to exist. Only the existence of a quasi-equilibrium can be proven. Several notions of irreducibility were proposed in order to allow a weakening of the survival assumption. We introduce an irreducibility condition which generalizes previous contributions. The irreducibility condition is necessary and sufficient in order to apply standard arguments for the transition from a quasi-equilibrium to a Walras equilibrium.

Without a non-satiation condition on the preferences, we show that our arguments remain valid in the case of dividend equilibria.

Sur des économies irréductibles

RÉSUMÉ. – Dans le modèle d’Arrow-Debreu, l'équilibre de Walras n'existe pas nécessairement sans une hypothèse de survie. Sans cette condition, seulement l'existence d'un quasi-équilibre est établi. Plusieurs concepts d'irréductibilité ont été alors proposés afin de permettre d'affaiblir l'hypothèse de survie. Nous introduisons une condition d'irréductibilité généralisant des contributions antérieures. Cette condition est nécessaire et suffisante pour le passage du quasi-équilibre à l'équilibre concurrentiel en appliquant des arguments standards.

Par ailleurs, en l'absence d'hypothèse de non-saturation sur les préférences, nous montrons que nos arguments restent valables dans le cas des équilibres concurrentiels avec dividendes.

* M. FLORIG : CERMSEM, Université de Paris 1.
I want to thank B. CORNET, M. FLORENZANO and an anonymous referee for helpful comments on previous versions of the paper. I am most grateful to J.M. BONNISSEAU for motivating this work and numerous helpful comments. Research was mainly carried out at the Universidad Carlos III Madrid and at the Universidad de Vigo. The support of these institutions is gratefully acknowledged.

1 Introduction

In the Arrow-Debreu model, the possibility of minimum-wealth situations at some prices may lead to non-existence of a Walras equilibrium. At a minimum-wealth situation, some consumers' income allows them to buy only the cheapest bundles of goods within their consumption set. Usually one prevents this situation from arising by an interiority condition, also called (strong) survival or Slater assumption. Loosely speaking, such a condition asserts that every consumer can consume some bundle of goods within the interior of his consumption set without exchanging anything. This is satisfied if every consumer's initial endowment is in the interior of his consumption set and if each firm can remain inactive (*cf.* ARROW and DEBREU [1954]). There are two aspects to such an assumption.

Firstly, the strong survival assumption means that every consumer has initially a positive amount of every existing commodity for trading. Already ARROW and DEBREU [1954] stated that "*[this assumption] is clearly unrealistic, and weakening is desirable*". Indeed, a lot of commodities are not even known by the consumers and most consumers have only one commodity to sell – their labour.

Secondly, should an agent have no share in any of the firms, then this condition can only hold if the consumption set has a non-empty interior. So every commodity must be consumable by every consumer. This as well is rather unrealistic. For example, industrial inputs are not all consumable by individuals. Should a spatial interpretation be given to the model, then the interiority condition implies that consumers can consume at two places simultaneously.

A weak survival assumption of the type that each firm can remain inactive and that every consumer has an initial endowment in his consumption set would be by far more acceptable. Then, consumers could have a zero initial endowment for some commodities, and their consumption set could have a non-empty interior. There are however simple and economically meaningful examples where the Walras equilibrium does not exist because of the lack of the strong survival assumption (see, for example, GALE [1976]).

Several authors established sufficient existence conditions which are stronger than the weak survival assumption, but weaker than the strong survival assumption. Considering instead the interiority condition, some sort of connection between the agents *via* their preferences and initial endowments, GALE [1957, 1976] established the existence of a Walras equilibrium for linear exchange economies. He assumed that there are no two subgroups such that group 1 has commodities group 2 likes, but group 2 has no commodities group 1 likes. Furthermore, the weak survival assumption must hold. One calls then the economy irreducible. There is a steadily growing literature adapting this idea to the standard Arrow-Debreu model, establishing weaker conditions for the existence of a Walras equilibrium than those stated in ARROW and DEBREU [1954]. MCKENZIE [1981] commented on the theorem on the existence of a competitive equilibrium: "*Perhaps the most dramatic innovation since 1959 is the discovery that the survival assumption, [...] can be* *dispensed with, in the presence of other assumptions, in particular the presence of [the] Assumption [...] that the economy is irreducible*". The first one to adapt Gale's irreducibility condition to the standard case was McKenzie [1959, 1961, 1981, 1987] himself. Subsequent contributions include Arrow and Hahn [1971], Moore [1975], Bergstrom [1976], Spivak [1978], Florenzano [1981], Geistdoerfer-Florenzano [1982], Hammond [1993], Maxfield [1997].

Debreu [1962] proposed an auxiliary concept called quasi-equilibrium, existing under the weak survival assumption. If no minimum-wealth situations occur at the quasi-equilibrium price, then this quasi-equilibrium is a Walras equilibrium. Arrow and Hahn [1971] introduced a similar concept called compensated equilibrium. The standard approach consists then in establishing an irreducibility condition which ensures that minimum-wealth levels do not occur at any of the quasi-equilibria. An exception to this approach is the one adopted by Gale [1957, 1976], who relies heavily on the linear preferences he considers. Considering net-trade sets Hammond [1993] proposes an irreducibility condition which, under rather weak conditions, is necessary and sufficient for excluding quasi-equilibria where minimum-wealth situations may occur.

In the framework of a standard Arrow-Debreu economy with a productive sector, but without non-satiation, we propose an irreducibility condition, which will turn out to be necessary and sufficient to exclude quasi-equilibria with minimum-wealth situations. If we restrict our framework to net trade sets as in Hammond [1993], our condition does not reduce to his. The proposed condition establishes a condition weaker than the currently known conditions for the existence of a Walras equilibrium in the standard Arrow-Debreu model including production. In fact, it is the weakest possible condition excluding quasi-equilibria with minimum-wealth situations.

Without a non-satiation condition a Walras equilibrium may not exist. Whatever the price may be, some consumers might wish to consume a commodity bundle within the interior of their budget set. An equilibrium can then be established by allowing some consumers to spend more than the value of their initial endowment. Such an equilibrium is called dividend equilibrium or equilibrium with slack and its existence is proven under a strong survival assumption (*cf.* Drèze and Müller [1980], Makarov [1981], Aumann and Drèze [1986], Mas Colell [1992]). Existence of a "*quasi-dividend equilibrium*" under a weak survival assumption follows from Florig [1998b]. Our irreducibility condition entails that every quasi-dividend equilibrium is a dividend equilibrium, thereby establishing rather weak conditions ensuring the existence of a dividend equilibrium.

Moreover, the irreducibility condition might elucidate situations of non-existence of the Walras equilibrium (or of a dividend equilibrium). Indeed, apart finding the minimal conditions ensuring the existence of a competitive (dividend) equilibrium, there remains the important question of what happens in a competitive economy where no Walras (or dividend) equilibrium exists. The quasi or compensated equilibrium, although interesting as a technical tool, do not give a convincing solution. In fact, they need not even be individually rational. In order to give an economically meaningful solution for situations where the Walras (dividend) equilibrium does not exist, Gay [1978], Danilov and Sotskov [1990], Marakulin [1990], Mertens [1996], FLORIG [1998a, 1998b] introduce and study generalized equilibrium concepts. In these papers, existence is shown under a weak survival assumption and without a non-satiation assumption. Due to the fact that more complex notions of prices are used than in the standard case, in these approaches, satiation problems of the preferences are inevitable, even under a non-satiation condition on the preferences. Under the proposed irreducibility condition, a generalized equilibrium reduces to a dividend equilibrium, and if furthermore non-satiation of the agents preferences holds, then it reduces to a Walras equilibrium.

Finally, the different irreducibility concepts are often adapted to extensions of the standard Arrow-Debreu model, as a continuum of traders, infinite dimensional commodity spaces, overlapping generations models, incomplete markets,... A list of such contributions includes HILDENBRAND [1972], BALASKO, CASS and SHELL [1980], WILSON [1981], BURKE [1988], GEANAKOPLOS and POLEMARCHAKIS [1991], HAMMOND [1993], COLES and HAMMOND [1994], GOTTARDI and HENS [1996]. In such extensions, the present irreducibility condition could also help to establish more general existence results.

In section 2, we present the model, the irreducibility condition and our main results. In section 3, we give a survey of the main irreducibility conditions. The proofs are given in the appendix.

2 The Model and the Results

Let I, J and L be finite sets of m consumers, n firms and ℓ commodities. Every consumer i is characterized by his consumption set $X_i \subset R^\ell$, his initial endowment $\omega_i \in R^\ell$ and his strict preference correspondence $P_i : \prod_{i \in I} X_i \times \prod_{j \in J} Y_j \to X_i$. Each firm j is characterized by a production set $Y_j \subset R^\ell$. We denote by $Y = \sum_{j \in J} Y_j$ the aggregate production set. For all $(i, j) \in I \times J$, the real $\theta_{ij} \geqslant 0$ represents consumer i's share in firm j and for all $j \in J$, $\sum_{i \in I} \theta_{ij} = 1$.

An economy $\mathcal{E}$ is a collection

$$\mathcal{E} = ((X_i, P_i, \omega_i)_{i \in I}, (Y_j)_{j \in J}, (\theta_{ij})_{(ij) \in I \times J}).$$

We will denote the set of feasible consumption-production plans by

$$\mathcal{F} = \{(x, y) \in \prod_{i \in I} X_i \times \prod_{j \in J} Y_j \mid \sum_{i \in I} x_i = \sum_{j \in J} y_j + \sum_{i \in I} \omega_i\}.$$

An allocation $x \in \prod_{i \in I} X_i$ is feasible, if there exists $y \in \prod_{j \in J} Y_j$ such that $(x, y) \in \mathcal{F}$.

Given a set $C \subset R^\ell$, let $\text{co}C$ be the convex hull of C, let $\text{pos}C = \{\sum_{\nu=1}^{t} \lambda_\nu z_\nu \mid z_\nu \in C, \lambda_\nu \geqslant 0, t \geqslant 0\}$ be the positive hull of C and let $\text{span}C = \text{pos}C - \text{pos}C$ be the vector subspace of R^ℓ spanned by C. We will use the shortcut $\text{span}\mathcal{E}$ for $\text{span}(\sum_{i\in I} X_i - \sum_{i\in I}\omega_i - \sum_{j\in J} Y_j)$.

Given $(x,y) \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j$, let $I^+(x,y) = \{i \in I \mid P_i(x,y) \neq \emptyset\}$, let $Z^+(x,y) = \sum_{i\in I^+(x,y)}(X_i - x_i - \sum_{j\in J}\theta_{ij}(Y_j - y_j))$.

Given a price $p \in R^\ell$ and a revenue $r \in R$, we define the budget set of consumer $i \in I$ by $B_i(p,r) = \{x_i \in X_i \mid p \cdot x_i \leqslant r\}$.

We will use the following assumptions.

ASSUMPTION A: For all $i \in I$ and all $(x,y) \in \mathcal{F}$, if $P_i(x,y) \neq \emptyset$ then $x_i \in \overline{\text{co}P_i(x,y)}$.

ASSUMPTION B: For all $(x,y) \in \mathcal{F}$, $\text{span}Z^+(x,y) = \text{pos}Z^+(x,y)$.

ASSUMPTION C: For all $i \in I$ and for all $(x,y) \in \mathcal{F}$, if $\xi_i \in P_i(x,y)$ and $\zeta_i \in X_i$ then there exists $\lambda \in]0,1[$ such that $\lambda\xi_i + (1-\lambda)\zeta_i \in K$ with $K = (\text{pos}(\overline{P_i(x,y)} - x_i) + x_i) \cup (\text{pos}(\overline{P_i(x,y)} - \omega_i - \sum_{j\in J}\theta_{ij}y_j) + \omega_i + \sum_{j\in J}\theta_{ij}y_j)$.

ASSUMPTION D: For all $i \in I$, $0 \in \text{co}(X_i - \omega_i - \sum_{j\in J}\theta_{ij}Y_j)$.

ASSUMPTION E: For all $(x,y) \in \mathcal{F}$ with $I^+(x,y) \neq \emptyset$,

$$\text{span}(Y - \sum_{j\in J} y_j) = \text{span}(\sum_{j\in J}\sum_{i\in I^+(x,y)}\theta_{ij}(Y_j - y_j)).$$

Assumption A excludes "*thick*" indifference surfaces, apart at satiation points. It could be weakened, but then one would need to work with augmented preference relations *à la* GALE and MAS-COLELL [1975].

Assumption B is a global interiority condition. Loosely speaking, it states that at any feasible allocation, the group of non-satiated agents are initially endowed (or they can produce) any of the goods they may consume. It ensures that if some quasi-equilibrium price is not orthogonal to $\text{span}Z^+(x,y)$, then at least one of the non-satiated consumers is not at a minimum-wealth level. If all agents have insatiable preferences, then this assumption is hardly restrictive. It then restricts the economy to the goods which actually exist or which could be produced in the considered economy. If it would not hold, we would need a market price, even for non existing and non producible commodities and consumers could formulate a demand for such goods.

Assumption C is fulfilled, for example, if the sets X_i are convex and, if the preference correspondences have relatively open images. It is also implied by assumptions in models with indivisible commodities such as overriding desirability of a divisible good (BROOME [1972], see also MAS-COLELL [1977], KHAN and YAMAZAKI [1981], HAMMOND [1993]).

Assumption D is a weak survival assumption. It avoids that agents may have an empty budget set.

Assumption E means that if a firm can either produce or dispose of a certain commodity, then at least one of the firms in which the non-satiated consumers have shares can either produce or dispose of this good. This assumption is in particular implied by a free-disposal assumption – however, it is much weaker. It holds trivially, if the considered economy is an exchange economy.

DEFINITION 2.1: A quasi-dividend equilibrium of $\mathcal{E}$ is an element $(x,y,p,r) \in \mathcal{F} \times R^{\ell} \times R^m$ such that

(a) for all $i \in I$, $x_i \in B_i(p,r_i)$ and $p \cdot (\omega_i + \sum_{j \in J} \theta_{ij} y_j) \leqslant r_i$;

(b) for all $i \in I$, $P_i(x,y) \cap B_i(p,r_i) \neq \emptyset$ implies $r_i = \inf p \cdot X_i$;

(c) for all $j \in J$ and for all $y'_j \in Y_j$, $p \cdot y'_j \leqslant p \cdot y_j$.

This definition reduces to the definition of a quasi-equilibrium, if we impose for all $i \in I$, $r_i = p \cdot (\omega_i + \sum_{j \in J} \theta_{ij} y_j)$. If all consumers have locally insatiable preferences, then every quasi-dividend equilibrium is a quasi-equilibrium.

DEFINITION 2.2: A dividend equilibrium of $\mathcal{E}$ is an element $(x,y,p,r) \in \mathcal{F} \times R^{\ell} \times R^m$ such that

(a) for all $i \in I$, $x_i \in B_i(p,r_i)$ and $p \cdot (\omega_i + \sum_{j \in J} \theta_{ij} y_j) \leqslant r_i$;

(b) for all $i \in I$, $P_i(x,y) \cap B_i(p,r_i) = \emptyset$;

(c) for all $j \in J$ and for all $y'_j \in Y_j$, $p \cdot y'_j \leqslant p \cdot y_j$.

This definition reduces to the definition of a Walras equilibrium, if we impose for all $i \in I$, $r_i = p \cdot (\omega_i + \sum_{j \in J} \theta_{ij} y_j)$. Note that, if all consumers have locally insatiable preferences, then every dividend equilibrium is a Walras equilibrium.

It is easy to see that every dividend equilibrium is a quasi-dividend equilibrium. A quasi-dividend equilibrium (x,y,p,r) is a dividend equilibrium, if the following condition holds:

$$\inf p \cdot X_i < \max\{p \cdot x_i, p \cdot (\omega_i + \sum_{j \in J} \theta_{ij} y_j)\}, \forall i \in I^+(x,y).$$

This condition is usually ensured by an assumption of the type: for all j in J, $0 \in Y_j$ and for all i in I, ω_i is in the interior of X_i. Especially, the latter one, being a quite strong assumption, there exist various notions of irreducibility of an economy, ensuring existence of a Walras equilibrium under less strong assumptions. As far as we know an adaptation to economies with possible satiation has not yet been proposed in literature.

DEFINITION 2.3: An economy $\mathcal{E}$ is irreducible at $(x,y) \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j$, if for every non-trivial partition (I_1, I_2) of $I^+(x,y)$ there exists $x' \in \prod_{i\in I} \mathrm{co}X_i$ and a system of $2m$ numbers $\lambda_i \geqslant 0$, $i = 1,...,m$, $\mu_i \geqslant 0$, $i = 1,...,m$ such that

(1) $i \in I^+(x,y)$ if and only if $\lambda_i + \mu_i > 0$;

(2) $x'_i \in \overline{\mathrm{co}P_i(x,y)}$, $\forall i \in I_1$ and $\exists i \in I_1$, $x'_i \in \mathrm{co}P_i(x,y)$;

(3) $\sum_{i\in I}\lambda_i(x'_i - \omega_i - \sum_{j\in J}\theta_{ij}y_j) + \sum_{i\in I}\mu_i(x'_i - x_i) \in \mathrm{co}(Y - \sum_{j\in J} y_j)$.

Note that in this condition, only directions matter and not magnitudes. It is easier to grasp the economic meaning of this condition in the case where all sets are convex. Moreover, assume in a first step that the considered economy is an exchange economy (*i.e.* $Y = 0$, or $Y = -R^\ell_+$ in the case of free disposal). There are two aspects to the condition. If we impose $\lambda_i = 0$ for all $i \in I$, then condition (3) asserts that a ponderated average of the prescribed change in the consumption plans is feasible. If we impose $\mu_i = 0$ for all $i \in I$, then this definition reduces to BERGSTROM'S [1976] irreducibility criterion. It then asserts that it is possible to find a change in the individual excess demands which is preferred by some agents and such that a ponderated average of these new excess demands is feasible, *i.e.* is equal to zero. The whole irreducibility criterion is a mixture of both aspects. When production is involved, then the considered ponderated changes give a feasible direction of change in the aggregate production plan. In the non-convex case one may think of some appropriate lotteries choosing the changes in consumption and production and which yield ponderated directions of change which are feasible in expectation.

LEMMA 2.1: (i) Let $p \in R^\ell$ such that $\mathrm{proj}_{\mathrm{span}\mathcal{E}}(p) = 0$ and let $(x,y) \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j$. Then, $\mathrm{proj}_{\mathrm{span}Z^+(x,y)}(p) = 0$.

(ii) If $(x,y) \in \mathcal{F}$ and $P_i(x,y) \neq \emptyset$ for all $i \in I$, then $\mathrm{span}Z^+(x,y) = \mathrm{span}\,\mathcal{E}$.

Note that, if $(x,y,p) \in \mathcal{F} \times R^\ell$ such that $\mathrm{proj}_{\mathrm{span}Z^+(x,y)}(p) = 0$ then for all $i \in I^+(x,y)$, $B_i(p, p \cdot x_i) = X_i$. At a dividend equilibrium (and at a Walras equilibrium) $p \cdot x_i \leqslant r_i$. Thus, if $I^+(x,y) \neq \emptyset$, this situation cannot be a dividend or a Walras equilibrium. If $I^+(x,y) = \emptyset$, then there are no scarce commodities in the economy, 0 is a Walras equilibrium price.

So we will be interested in quasi-(dividend) equilibria such that $\mathrm{proj}_{\mathrm{span}Z^+(x,y)}(p) \neq 0$. By Lemma 2.1. (i), this is slightly more demanding than $\mathrm{proj}_{\mathrm{span}\mathcal{E}}(p) \neq 0$. It would not be very restrictive to impose

$\operatorname{span} Z^+(x,y) = \operatorname{span} \mathcal{E}$. This means that the consumers in $I^+(x,y)$ could consume, produce or dispose of the same commodities as I (although they may have not enough resources to actually do so). It is for example implied by the free disposal assumption.

Here, we are not concerned with the existence of a quasi-dividend equilibrium. General theorems for the existence of quasi-equilibria can be found, for example, in DEBREU [1962], ARROW-HAHN [1971] and FLORENZANO [1981]. Existence of a quasi-dividend equilibrium follows from FLORIG [1998b]. We concentrate on the transition from the quasi-dividend equilibrium to the dividend equilibrium.

PROPOSITION 2.1: Let $\mathcal{E}$ be an economy such that Assumptions A and B hold, and let (x,y,p,r) be a quasi-dividend equilibrium with $\operatorname{proj}_{\operatorname{span} Z^+(x,y)}(p) \neq 0$. If $\mathcal{E}$ is irreducible at (x,y), then for all $i \in I^+(x,y)$, $\inf p \cdot X_i < r_i$ and therefore (x,y,p,r) is a dividend equilibrium.

As a corollary we may deduce that under the assumptions of Proposition 2.1., every quasi-equilibrium is a Walras equilibrium.

PROPOSITION 2.2: Let $\mathcal{E}$ be an economy satisfying Assumptions C and D, let $(x,y) \in \mathcal{F}$. If $\mathcal{E}$ is not irreducible at (x,y), then there exists $p \in R^\ell$ and $r \in R^m$ such that (x,y,p,r) is a quasi-dividend equilibrium with $\operatorname{proj}_{\operatorname{span}\mathcal{E}}(p) \neq 0$ and $\{i \in I \mid p \cdot x_i = r_i = \inf p \cdot X_i\} \cap I^+(x,y) \neq \emptyset$.
If moreover, Assumption E holds, then $\operatorname{proj}_{\operatorname{span} Z^+(x,y)}(p) \neq 0$.

Thus, irreducibility is also necessary in order to exclude the existence of quasi-dividend equilibria with minimum-wealth situations. If all consumers have locally insatiable preferences, then a quasi-dividend equilibrium is a quasi-equilibrium. Therefore, the present irreducibility condition is also necessary in order to exclude the existence of quasi-equilibria with minimum-wealth situations.

COROLLARY 2.1: Let $\mathcal{E}$ be an economy such that Assumptions A - E hold. Then, every, quasi-dividend equilibrium with $\operatorname{proj}_{\operatorname{span}\mathcal{E}}(p) \neq 0$ is a dividend equilibrium if and only if $\mathcal{E}$ is irreducible at every $(x,y) \in \mathcal{F}$.

Hence, the proposed irreducibility condition is necessary and sufficient for the standard arguments allowing for the transition from the quasi-dividend equilibrium (resp. quasi-equilibrium) to the dividend (resp. Walras) equilibrium.

Note that neither the assumptions used, nor the proposed irreducibility condition imply that consumption or production sets are convex, not even that all commodities are perfectly divisible. Therefore, the present approach could also be used in the presence of indivisible commodities.

3 Links with other Irreducibility Conditions

In this section, we discuss briefly the main irreducibility conditions which serve to exclude minimum-wealth situations. We will not discuss GALE's [1976] irreducibility condition. Considering linear utility functions, he follows a different approach in order to prove the existence of a Walras equilibrium, relying on an induction argument which uses extensively the linear structure of his model. The following irreducibility conditions have all in common that they are sufficient in order to establish that minimum-wealth situations do not occur at a quasi-equilibrium. We will see that they are all special cases of the irreducibility condition presented in section 2.

We denote by $\{e^1,\dots,e^\ell\}$ the canonical basis of R^ℓ. We posit the following assumption throughout this section. It is stated for simplicity and it could be weakened at the cost of more complex notations.

ASSUMPTION F: For all $(x,y) \in \mathcal{F}$, $I^+(x,y) = I$; for all $i \in I$, P_i is convex valued, X_i is convex; pos$\mathcal{E} = \mathcal{R}^\ell$; $Y - R^\ell_+ \subset Y$; and Y is convex.

The first condition is the strong survival assumption. It is a generalization of a condition in Theorem 1 in ARROW-DEBREU [1954]. It is nowadays the standard condition in order to exclude minimum-wealth levels. It is easy to check that it is sufficient in order to exclude quasi-equilibria with non-zero prices and minimum-wealth levels.

DEFINITION 3.1: An economy $\mathcal{E}$ satisfies the strong survival condition, if for all $i \in I$, $\text{pos}(X_i - \omega_i - \sum_{j \in J} \theta_{ij} Y_j) = R^\ell$.

The next condition is a generalization of the irreducibility condition in Theorem 2 in ARROW-DEBREU [1954]. It replaces the interiority condition by a stronger condition on the preferences.

DEFINITION 3.2: An economy $\mathcal{E}$ satisfies the desirability condition, if

$$\mathcal{D} = \{h \in L \mid \forall (x,y) \in \mathcal{F}, \forall i \in I, \exists \varepsilon > 0, x_i + \varepsilon e^h \in P_i(x,y)\} \neq \emptyset$$

and if for all $i \in I$, there exists $h \in \mathcal{D} \cup \mathcal{P}$ such that

$$-e^h \in \text{pos}(X_i - \omega_i - \textstyle\sum_{j \in J} \theta_{ij} Y_j)$$

with

$$\mathcal{P} = \{h \in L \mid \forall (x,y) \in \mathcal{F}, \exists z \in Y, z_{h'} \geqslant y_{h'} \forall h' \in L \setminus \{h\}, \exists h'' \in \mathcal{D}, z_{h''} > y_{h''}\}.$$

The set $\mathcal{D}$ are the commodities which are always desired by everybody. The set $\mathcal{P}$ are the commodities which have always a strictly positive marginal productivity in order to produce commodities in $\mathcal{D}$. The desirability condition asserts that every consumer is initially endowed with some commodity which is either desired by everybody or which has a strictly positive marginal productivity in order to produce commodities which are always desired by everybody.

DEFINITION 3.3: An economy $\mathcal{E}$ is Bergstrom irreducible at $(x,y) \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j$, if for every partition of I into two non-empty subsets (I_1,I_2), there exists $x' \in \prod_{i\in I} X_i$ and a system of m numbers $\lambda_i > 0$, $i = 1,...,m$, such that

(1) $x'_i \in \overline{P_i(x,y)}$, $\forall i \in I_1$ and $\exists i \in I_1$, $x'_i \in P_i(x,y)$;

(2) $\sum_{i\in I} \lambda_i(x'_i - \omega_i - \sum_{j\in J} \theta_{ij} y_j) \in (Y - \sum_{j\in J} y_j)$.

Irreducibility reduces to Bergstrom irreducibility by imposing $\mu_i = 0$ for all $i \in I$.

DEFINITION 3.4: An economy $\mathcal{E}$ is McKenzie-Debreu irreducible at $(x,y) \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j$, if for every partition of I into two non-empty subsets (I_1,I_2), there exists $x' \in \prod_{i\in I} X_i$, such that

(1) $x'_i \in \overline{P_i(x,y)}$, $\forall i \in I_1$ and $\exists i \in I_1$, $x'_i \in P_i(x,y)$;

(2) $\sum_{i\in I}(x'_i - \omega_i) - \sum_{i\in I_2}(\omega_i - x_i) \in Y$.

In other terms, it is possible to distribute $\omega + \sum_{i\in I_2}(\omega_i - x_i) + y'$, $y' \in Y$ amongst the consumers, making group I_1 better off according to (1).

DEFINITION 3.5: An economy $\mathcal{E}$ is Arrow-Hahn irreducible at $(x,y) \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j$, if for every partition of I into two non-empty subsets (I_1,I_2), there exists an allocation $x' \in \prod_{i\in I} X_i$, $\omega' \in R^\ell$ such that

(1) $x'_i \in \overline{P_i(x,y)}$, $\forall i \in I$ and $\exists i_1 \in I_1$, $x'_{i_1} \in P_i(x,y)$;

(2) $\sum_{i\in I} x'_i \in \{\omega'\} + Y$ and $\forall k \in L$ with $\omega'^k > \omega^k$ there exists $\lambda^k > 0$ such that $\sum_{i\in I_2} \omega_i - \lambda^k e^k \in \sum_{i\in I_2} X_i + R^\ell_+$.

An economy $\mathcal{E}$ is Arrow-Hahn irreducible, if for any feasible allocation x and any proper, non-empty subset of consumers I_1, there exists an allocation x', making everybody better off and some $i_1 \in I_1$ strictly better off. Moreover, x' would be feasible if the total initial endowment were ω' and if for every good k such that ω' increases the total initial endowment of this good, the group I_2 could consume less than their initial endowment in k.

Without the assumption that for every $j \in J$, $0 \in Y_j$, neither McKenzie-Debreu, nor Arrow-Hahn irreducibility imply that a quasi-equilibrium is a

Walras equilibrium (*cf.* FLORIG [1997]). Thus, McKenzie-Debreu and Arrow-Hahn irreducibility are only applicable when $0 \in Y_j$ for every $j \in J$.

The next irreducibility condition is due to HAMMOND [1993].

DEFINITION 3.6: An exchange economy $\mathcal{E}$ is Hammond irreducible at $x \in \prod_{i \in I} X_i$, if for every non-trivial partition (I_1, I_2) of I there exists $x' \in \prod_{i \in I} X_i$ and a system of m numbers $\lambda_i \in [0,1]$, $i = 1,...,m$, such that

(1) $x'_i \in \overline{P_i(x)}$, $\forall i \in I_1$ and $\exists i \in I_1$, $x'_i \in P_i(x)$;

(2) $\sum_{i \in I} \lambda_i (x'_i - \omega_i) + \sum_{i \in I} (1 - \lambda_i)(x'_i - x_i) + \sum_{i \in I_2} (x_i - \omega_i) = 0$.

The difference with our irreducibility condition is the additional term $\sum_{i \in I_2} (x_i - \omega_i)$ in Hammond's irreducibility condition. Without any condition on the considered exchange economy neither seems to imply the other. However, under Assumptions A - F (and $J = \emptyset$) Hammond irreducibility at every feasible point is equivalent to the non-existence of a quasi-equilibrium with minimum-wealth levels just as our irreducibility condition. In order to understand why both conditions work equally well one has to remind how one usually proves that at a quasi-equilibrium no consumer is at minimal-wealth. The irreducibility conditions are applied to a quasi-equilibrium. The set of consumers at minimal-wealth is supposed to be I_2. Assuming that I_2 is non-empty one proceeds by contraposition. One computes the scalar product between quasi-equilibrium price and the left hand side of 3.6. (2) (resp. 2.3. (3)). The scalar product between quasi equilibrium price and $\sum_{i \in I_2} (x_i - \omega_i)$ will be equal to zero and therefore this term plays no role whatsoever in establishing a contradiction.

PROPOSITION 3.1: Let $\mathcal{E}$ be an economy such that Assumptions C-F hold and suppose one of the following:

(i) the strong survival condition holds;

(ii) the desirability condition holds;

(iii) at every $(x,y) \in \mathcal{F}$, $\mathcal{E}$ is Bergstrom irreducible;

(iv) $J = \emptyset$ and at every $(x,y) \in \mathcal{F}$, $\mathcal{E}$ is Hammond irreducible

(v) for every $j \in J$, $0 \in Y_j$ and at every $(x,y) \in \mathcal{F}$, $\mathcal{E}$ is either McKenzie-Debreu or Arrow-Hahn irreducible.

Then, the economy is irreducible at every $(x,y) \in \mathcal{F}$.

PROOF of PROPOSITION 3.1: Suppose the economy is not irreducible at some $(x,y) \in \mathcal{F}$. Then there exists a quasi-dividend equilibrium (x,y,p,r) with $p \neq 0$ and such that the set I_2 of consumers at minimum-wealth level is non-empty. By Assumption F this is also a quasi-equilibrium.

(i) The strong survival assumption implies the existence of

$$(\xi, \upsilon) \in \prod_{i \in I} X_i \times \prod_{j \in J} Y_j$$

such that for all $i \in I$, $p \cdot (\xi_i - \omega_i - \sum_{j \in J} \theta_{ij} \upsilon_j) < 0$. By profit maximization $p \cdot (\xi_i - \omega_i - \sum_{j \in J} \theta_{ij} y_j) \leqslant p \cdot (\xi_i - \omega_i - \sum_{j \in J} \theta_{ij} \upsilon_j) < 0$.

Since (x, y, p) is a quasi-equilibrium, for all $i \in I$,

$$p \cdot (x_i - \omega_i - \sum_{j \in J} \theta_{ij} y_j) = 0.$$

Thus $I_2 = \emptyset$ yielding a contradiction.

(ii) By Assumption F, there is at least one consumer with a non-minimal income. This implies that all commodities in $\mathcal{D}$ have a strictly positive price. This implies that all commodities in $\mathcal{P}$ have a strictly positive price. Then desirability implies that all consumers have non-minimal wealth yielding a contradiction.

(iii) - (v) These conditions exclude the existence of quasi-equilibria with minimum-wealth levels and a non-zero price (*cf.* McKenzie [1959, 1961], Arrow and Hahn [1971], Bergstrom [1976], Florenzano [1981], Hammond [1993]). This yields a contradiction.

From Florenzano [1981, 1982], we know that Bergstrom irreducibility is, in the case of an exchange economy, implied by McKenzie-Debreu or Arrow-Hahn irreducibility. Moreover, from Geistdoerfer-Florenzano [1982], a Bergstrom irreducible economy need not be, neither McKenzie-Debreu nor Arrow-Hahn irreducible. In the production case, following Florenzano's [1981] arguments closely, one may prove that Bergstrom irreducibility is implied by McKenzie-Debreu or Arrow-Hahn irreducibility (Florig [1997]).

Proposition 3.2: Let $\mathcal{E}$ such that Assumption F holds, for all $j \in J$, $0 \in Y_j$ and Y is convex. Suppose $\mathcal{E}$ is either McKenzie-Debreu or Arrow-Hahn irreducible at $(x, y) \in \prod_{i \in I} X_i \times \prod_{j \in J} Y_j$, then it is also Bergstrom irreducible at (x, y) and hence it is also irreducible at (x, y). ■

• References

ARROW K.J., DEBREU G. (1954). – « Existence of an Equilibrium for a Competitive Market », *Econometrica*, 22, pp. 265-290.

ARROW K.J., HAHN F.H. (1971). – *General Competitivity Analysis*, Holden-Day, San Francisco.

AUMANN R.J., DRÈZE J.H. (1986). – « Values of Markets with Satiation or Fixed Prices », *Econometrica*, 54 (6), pp. 1271-1318.

BALASKO Y., CASS D., SHELL K. (1980). – « Existence of Competitive Equilibrium in a General Overlapping Generations Model », *Journal of Economic Theory*, 23, pp. 307-322.

BERGSTROM T. (1976). – « How to Discard Free Disposability – at no Cost », *Journal of Mathematical Economics*, 3, pp. 131-134.

BROOME J. (1972). – « Existence of Equilibrium in Economies with Indivisible Commodities », *Journal of Economic Theory*, 5, pp. 224-250.

BURKE J. (1988). – « On the Existence of Price Equilibria in Dynamic Economies », *Journal of Economic Theory*, 44, pp. 281-300.

COLES J.L., HAMMOND P.J. (1995). – « Walrasian Equilibrium without Survival: Existence, Efficiency and Remedial Policy », in *Choice Welfare and Development – A Festschrift in Honour of Amartya K. Sen*, edited by K. Basu, P. Pattanaik, K. Suzumara, Clarendon Press, Oxford, pp. 32-64.

DANILOV V.I., SOTSKOV A.I. (1990). – « A Generalized Economic Equilibrium », *Journal of Mathematical Economics*, 19, pp. 341-356.

DEBREU G. (1962). – « New Concepts and Techniques for Equilibrium Analysis », *International Economic Review*, 3, pp. 257-273.

DRÈZE J., MÜLLER H. (1980). – « Optimality Properties of Rationing Schemes », *Journal of Economic Theory*, 23, pp. 131-149.

FLORENZANO M. (1981). – *L'équilibre économique général transitif et intransitif: Problèmes d'Existence*, CNRS, Paris.

FLORIG M. (1997). – « Contributions à la théorie de l'équilibre général », Ph.D. thesis, Université de Paris I.

FLORIG M. (1998a). – « A Note on Different Concepts of Generalized Equilibria », *Journal of Mathematical Economics*, 29(3), pp. 245-254.

FLORIG M. (1998b). – « Hierarchic Competitive Equilibria », *Cahier Eco&Math*, Université de Paris I.

GALE D. (1957). – « Price Equilibrium for Linear Models of Exchange », *Technical Report* P-1156, The Rand Corporation.

GALE D. (1976). – « The Linear Exchange Model », *Journal of Mathematical Economics*, 3, pp. 205-209.

GALE D., MAS-COLELL A. (1975). – « An Equilibrium Existence Theorem for a General Model without Ordered Preferences », *Journal of Mathematical Economics*, 2, pp. 9-15.

GAY A. (1978). – « The Exchange Rate Approach to General Economic Equilibrium », *Economie Appliquée*, Archives de l'ISMEA XXXI, nos 1 2, pp. 159-174.

GEANAKOPLOS J.D., POLEMARCHAKIS H.M. (1986). – Existence, Regularity, and Constrained Suboptimality of Competitive Allocations when Markets are Complete, in *Uncertainty, Information and Communication. Essays in Honour of Kenneth Arrow*, (W.P. Heller, R.M. Ross, and D.A. Starrett, Eds.), Vol. 3., Cambridge University Press, Cambridge, U.K.

GEANAKOPLOS J.D., POLEMARCHAKIS H.M. (1991). – Overlapping Generations, in *Handbook of Mathematical Economics*, Volume IV, Edited by W. Hildenbrand and H. Sonnenschein., Elsevier Science Publishers.

GEISTDOERFER-FLORENZANO M. (1982). – « The Gale-Nikaido-Debreu Lemma and the Existence of Transitive Equilibrium with or without the Free-Disposal Assumption », *Journal of Mathematical Economics*, 9, pp. 113-143.

GOTTARDI P, HENS T. (1996). – « The Survival Assumption and Existence of Competitive Equilibria When Asset Markets are Incomplete », *Journal of Economic Theory*, 71, pp. 313-323.

HAMMOND P.J. (1993). – Irreducibility, Resource Relatedness and Survival in Equilibrium with Individual Nonconvexities in *General Equilibrium, Growth and Trade*, edited by R. Becker, M. Boldrin, R. Jones, and W. Thomson, San Diego, Academic Press, pp. 73-115.

HILDENBRAND W. (1972). – Metric Measure Spaces of Economic Agents, in *Proceedings of the Sixth Berkley Symposium on Mathematical Statistics and Probability*, Vol. II, edited by L. Le Cam, J. Neyman, and E.L. Scott, University of California Press,m Berkeley, pp. 81-95.

KHAN M.A., YAMAZAKI A. (1981). – « On the Cores of Economies with Indivisible Commodities and a Continuum of Traders », *Journal of Economic Theory*, 24, pp. 218-255.

MAKAROV V.L. (1981). – « Some Results on General Assumptions about the Existence of Economic Equilibrium », *Journal of Mathematical Economics*, 8, pp. 87-100.

MARAKULIN V. (1990). – Equilibrium with Nonstandard Prices in Exchange Economies, in *Optimal Decisions in Market and Planned Economies*, edited by R. Quandt and D. Triska, Westview Press, London, pp. 268-282.

MAS-COLELL A. (1977). – « Indivisible Commodities and General Equilibrium Theory », *Journal of Economic Theory*, 16, pp. 443-456.

MAS-COLELL A. (1992). – Equilibrium Theory with Possibly Satiated Preferences, in *Equilibrium and Dynamics: Essays in Honour of David Gale*, edited by M. Majumdar, Macmillan, London, pp. 201-213.

MAXFIELD R.R. (1997). – « General Equilibrium and the Theory of Directed Graphs », *Journal of Mathematical Economics*, 27, pp. 23-51.

MCKENZIE L. (1959). – « On the Existence of General Equilibrium for a Competitive Market, », *Econometrica*, 27, pp. 54-71.

MCKENZIE L. (1961). – « On the Existence of General Equilibrium for a Competitive Market: Some Corrections », *Econometrica*, 29(2), pp. 247-248.

MCKENZIE L. (1981). – « The Classical Theorem on Existence of Competitive Equilibrium », *Econometrica*, 29, 2, pp. 247-248.

MCKENZIE L. (1987). – General Equilibrium, in *The New Palgrave: A Dictionary of Economics*, edited by J. Eatwell, M. Milgate and P. Newman, Macmillan, London.

MERTENS J.F. (1996). – « The Limit-Price Mechanism », CORE *Discussion Paper*, 9650, forthcoming in *Essays in Honour of Martin Shubik* edited by P. Dubey and J. Geanakoplos.

SPIVAK A. (1978). – « A note on Arrow-Hahn's Resource Relatedness (or McKenzie's Irreducibility) », *International Economic Review*, 19, pp. 527-531.

WILSON C.A. (1981). – « Equilibrium in Dynamic Models with an Infinity of Agents », *Journal of Economic Theory*, 24, pp. 95-111.

APPENDIX

PROOF of LEMMA 2.1. (i) If $\mathrm{proj}_{\mathrm{span}\mathcal{E}}(p) = 0$, then for all $i \in I$, for all $\xi_i \in X_i$, $p \cdot (\xi_i - x_i) = 0$ and for all $j \in J$, for all $\upsilon_j \in Y_j$, $p \cdot (\upsilon_j - y_j) = 0$. Thus, for all $z \in Z^+(x,y)$, $p \cdot z = 0$ and therefore $\mathrm{proj}_{\mathrm{span}Z^+(x,y)}(p) = 0$.

(ii) If $(x,y) \in \mathcal{F}$ and $P_i(x,y) \neq \emptyset$ for all $i \in I$, then

$$\begin{aligned}\textstyle\sum_{i\in I} X_i - Y - \sum_{i\in I}\omega_i &= \textstyle\sum_{i\in I} X_i - Y - \sum_{i\in I} x_i \\ &\quad \textstyle+ \sum_{j\in J} y_j \subset \sum_{i\in I}(X_i - x_i) - \sum_{j\in J}\sum_{i\in I}\theta_{ij}(Y_j - y_j) \\ &= \textstyle\sum_{i\in I}(X_i - x_i - \sum_{j\in J}\theta_{ij}(Y_j - y_j)) = Z^+(x,y)\end{aligned}$$

and moreover, $Z^+(x,y) \subset \mathrm{co}(\sum_{i\in I} X_i - Y - \sum_{i\in I}\omega_i)$. Thus,

$$\mathrm{span}Z^+(x,y) = \mathrm{span}(\textstyle\sum_{i\in I} X_i - Y - \sum_{i\in I}\omega_i). \qquad \blacksquare$$

PROOF of PROPOSITION 2.1. Let $(I_1, I_2,)$ be a partition of $I^+(x,y)$ such that $I_1 = \{i \in I^+(x,y) \mid \inf p \cdot X_i < r_i\}$. Let $I_2 = I^+(x,y) \setminus I_1$. It will be sufficient to prove that $I_1 = I^+(x,y)$.

Suppose first that $I_1 = \emptyset$. Hence,

$$\begin{aligned}\inf \textstyle\sum_{i\in I^+(x,y)} p \cdot X_i &= \textstyle\sum_{i\in I^+(x,y)} p \cdot x_i \geqslant \sum_{i\in I^+(x,y)} p \cdot (\omega_i + \sum_{j\in J}\theta_{ij} y_j) \\ &= \sup \textstyle\sum_{i\in I^+(x,y)} p \cdot (\omega_i + \sum_{j\in J}\theta_{ij} Y_j).\end{aligned}$$

Thus, $\mathrm{pos}Z^+(x,y) \subset \{z \in R^\ell \mid 0 \leqslant p \cdot z\}$ and

$$-\mathrm{pos}Z^+(x,y) \subset \{z \in R^\ell \mid 0 \geqslant p \cdot z\}.$$

By Assumption B, $\mathrm{pos}Z^+(x,y) = -\mathrm{pos}Z^+(x,y) = \mathrm{span}Z^+(x,y)$ and thus $\mathrm{span}Z^+(x,y) \subset \{z \in R^\ell \mid 0 = p \cdot z\}$ and $\mathrm{proj}_{\mathrm{span}Z^+(x,y)} p = 0$, a contradiction. Therefore $I_1 \neq \emptyset$.

We will now prove that $I_2 = \emptyset$. If (I_1, I_2) were a non-trivial partition of $I^+(x,y)$, then, since $\mathcal{E}$ is irreducible at (x,y), there exists $x' \in \prod_{i\in I}\mathrm{co}X_i$ and $2m$ real numbers $\lambda_i \geqslant 0$, $i = 1,\ldots,m$, $\mu_i \geqslant 0$, $i = 1,\ldots,m$, satisfying the relations (1) - (3) of Definition 2.2 with respect to (x,y) and (I_1, I_2). By Assumption A, for every $i \in I_1$ such that $x'_i \in \overline{\mathrm{co}P_i(x,y)}$, $p \cdot x'_i - r_i \geqslant 0$ and for $i \in I_1$ such that $x'_i \in \mathrm{co}P_i(x,y)$, $p \cdot x'_i - r_i > 0$. Hence,

$$p \cdot (\sum_{i\in I_1}\lambda_i(x'_i - \omega_i - \sum_{j\in J}\theta_{ij} y_j) + \sum_{i\in I_1}\mu_i(x'_i - x_i)) > 0.$$

As $p \in N_Y(\sum_{j\in J} y_j)$– the normal cone of Y at $\sum_{j\in J} y_j \in Y$,

$$p \cdot (\sum_{i\in I}\lambda_i(x'_i - \omega_i + \sum_{j\in J}\theta_{ij} y_j) + \sum_{i\in I}\mu_i(x'_i - x_i)) \leqslant 0.$$

Thus,

$$p \cdot (\sum_{i \in I_2} \lambda_i (x'_i - \omega_i + \sum_{j \in J} \theta_{ij} y_j) + \sum_{i \in I_2} \mu_i (x'_i - x_i)) < 0.$$

Consequently, there exists $i \in I_2$ such that either

$$p \cdot (x'_i - \omega_i + \sum_{j \in J} \theta_{ij} y_j) < 0$$

or $p \cdot (x'_i - x_i) < 0$. This is not possible by the definition of (I_1, I_2). Therefore, $I_1 = I^+(x,y)$. ■

PROOF of PROPOSITION 2.2. The economy is not irreducible at (x,y). Thus, there exists a non-trivial partition (I_1, I_2) of $I^+(x,y)$, such that for every $i \in I_1$, for every $2m$ numbers $\lambda_i \geqslant 0, i = 1,...,m,$ $\mu_i \geqslant 0, i = 1,...,m,$ satisfying condition (1) of Definition 2.3, for every $x' \in \prod_{i \in I} \mathrm{co} X_i$ with $x'_i \in \mathrm{co} P_i(x,y)$ for $i \in I_1$ we have

$$\sum_{i \in I} \lambda_i (x'_i - \omega_i - \sum_{j \in J} \theta_{ij} y_j) + \sum_{i \in I} \mu_i (x'_i - x_i) \notin \mathrm{co}(Y - \sum_{j \in J} y_j).$$

Therefore, $0 \notin C$ with $C =$

$$\sum_{i \in I_1} (\lambda_i + \mu_i) \mathrm{co} P_i(x,y) + \sum_{i \in I_2} (\lambda_i + \mu_i) \mathrm{co} X_i - \sum_{i \in I} \lambda_i (\omega_i + \sum_{j \in J} \theta_{ij} y_j) - \sum_{i \in I} \mu_i x_i - \mathrm{co}(Y - \sum_{j \in J} y_j).$$

The set C is convex. Thus, there exists a vector $p \in \mathrm{span} C \setminus \{0\}$ separating 0 and C such that for all $c \in \overline{C}$, $0 \leqslant p \cdot c$ and for some $c \in C$, $0 < p \cdot c$.

Then, for all $i \in I_2$, $(X_i - x_i) \subset \overline{C}$ and $(X_i - \omega_i - \sum_{j \in J} \theta_{ij} y_j) \subset \overline{C}$. This implies that for all $i \in I_2$, $p \cdot x_i = p \cdot (\omega_i - \sum_{j \in J} \theta_{ij} y_j) = \inf p \cdot X_i$.

Note that $-\mathrm{co}(Y - \sum_{j \in J} y_j) \in \overline{C}$. Thus, for all $y' \in \prod_{j \in J} Y_j$, $p \cdot \sum_{j \in J} y'_j \leqslant p \cdot \sum_{j \in J} y_j$ and therefore all firms are maximizing profit.

For all $i \in I$, let $r_i = \max\{p \cdot (\omega_i + \sum_{j \in J} \theta_{ij} y_j), p \cdot x_i\}$. We will now prove that for all $i \in I_1$ with $\inf p \cdot X_i < r_i$, $P_i(x,y) \cap B_i(p, r_i) = \emptyset$. Suppose this is not the case for some $i \in I_1$ with $\inf p \cdot X_i < r_i$, then there exists $\xi_i \in P_i(x,y) \cap B_i(p, r_i)$. We must have $p \cdot (\xi_i - x_i) \geqslant 0$ and $p \cdot (\xi_i - \omega_i - \sum_{j \in J} \theta_{ij} y_j) \geqslant 0$. Hence, $p \cdot \xi_i = r_i$. By a similar argument we have for every $z_i \in K$, where K is as defined in Assumption C, $p \cdot z_i \geqslant r_i$. Let $\zeta_i \in X_i$ such that $p \cdot \zeta_i < r_i$. Thus, $\zeta_i \in X_i \setminus K$. Now by Assumption C, there exist $\eta \in]0,1[$ such that $\eta \xi_i + (1-\eta)\zeta_i \in K$. Hence, $p \cdot (\eta \xi_i + (1-\eta)\zeta_i) \geqslant r_i$, contradicting $p \cdot \zeta_i < r_i$ together with $p \cdot \xi_i = r_i$.

It remains to prove that $\mathrm{proj}_{\mathrm{span}\mathcal{E}}(p) \neq 0$. One easily checks that since there exists $c \in C$ such that $0 < p \cdot c$, one of the following three must hold:

(a) there exists $j \in J, \upsilon_j \in Y_j$ such that $p \cdot (y_j - \upsilon_j) > 0$;

(b) there exists $i \in I^+(x,y), \xi_i \in X_i$ such that $p \cdot (\xi_i - x_i) > 0$;

(c) there exists $i \in I^+(x,y), \xi_i \in X_i$ such that

$$p \cdot (\xi_i - \omega_i - \textstyle\sum_{j \in J} \theta_{ij} y_j) > 0.$$

If (a) holds, then $\mathrm{proj}_{\mathrm{span}(\sum_{j\in J}(Y_j - y_j))}(p) \neq 0$ and $\mathrm{proj}_{\mathrm{span}\mathcal{E}}(p) \neq 0$. If moreover, Assumption E holds, then (a) implies that there exists $\gamma \in \prod_{j \in J} Y_j$, such that $p \cdot \sum_{i \in I^+(x,y)} \sum_{j \in J} \theta_{ij}(\gamma_j - y_j) \neq 0$. Hence, $\mathrm{proj}_{\mathrm{span}Z^+(x,y)}(p) \neq 0$. If (b) holds then $\mathrm{proj}_{\mathrm{span}Z^+(x,y)}(p) \neq 0$, implying by Lemma 2.1. (i) that $\mathrm{proj}_{\mathrm{span}\mathcal{E}}(p) \neq 0$. By Assumption D and the fact that firms maximize profit at y, for every $i \in I^+(x,y)$, there exists $\zeta_i \in X_i$, such that $p \cdot (\zeta_i - \omega_i - \sum_{j\in J} \theta_{ij} y_j) \leqslant 0$. Thus, if (c) holds, then there exists $i \in I^+(x,y)$ and $\xi_i, \zeta_i \in X_i$ such that $p \cdot (\xi_i - \zeta_i) > 0$. So either $p \cdot (\xi_i - x_i) \neq 0$ or $p \cdot (\zeta_i - x_i) \neq 0$. Therefore, $\mathrm{proj}_{\mathrm{span}Z^+(x,y)}(p) \neq 0$, implying by Lemma 2.1. (i) that $\mathrm{proj}_{\mathrm{span}\mathcal{E}}(p) \neq 0$. ■