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Continuity and Uniqueness of Equilibria for Linear Exchange Economies[1]

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Abstract. The purpose of this paper is to study the continuity and uniqueness properties of equilibria for linear exchange economies. We characterize the sets of utility vectors and initial endowments for which the equilibrium price is unique and respectively the set for which the equilibrium allocation is unique. We show that the equilibrium allocation correspondence is continuous with respect to the initial endowments and we characterize the set of full measure where the equilibrium allocation correspondence with respect to the initial endowments and utility vectors is continuous.

Key Words. General equilibrium, linear utility functions, sensitivity analysis, equilibrium correspondence.

1. Introduction

Linear exchange economies have been studied extensively (see Refs. 1–6). Economies with agents having preferences representable by linear utility functions constitute a basic model. The standard approach in the literature considers differentiable strictly quasiconcave utility functions with a boundary condition which, roughly speaking, means that the indifference curves do not cut the boundary of the consumption set. This implies that each commodity is necessary for each consumer whatever are the prices and his income. This is a relatively strong condition and the study of linear economies appear as a natural simple setting to remove both the strict quasiconcavity condition and the boundary condition.

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Linear economies are also interesting, since they generate strong results which are different from those obtained by the standard approach, and the simple parametrization of the utility functions allows us to consider them as variables whereas they are usually fixed. Moreover, on account of the strong properties of the linear model, it allows for gaining some insight and therefore is a valuable complement. The uniqueness of the equilibrium price vector under a strong survival assumption for only one consumer and the unique utility level at equilibrium (Ref. 3) are two examples of properties which cannot be obtained in the standard approach via assumptions involving only the fundamentals of the economy. These uniqueness properties can be used in order to define an equilibrium in the setting of imperfect competition models. For example, in Bonnisseau–Florig (Ref. 7), the uniqueness of equilibrium prices together with some of the results found in the present paper and its continuation (Ref. 8) are used in order to analyze oligopoly equilibria. Another important feature of the linear exchange model is its computability. Eaves (Ref. 4) proposes an algorithm which computes in a finite number of steps either an equilibrium or a reduction of the economy if no equilibrium exists.

Moreover, linear exchange economies turn out to be very useful for different models where the agents' preferences are not representable by linear utility functions, but rather by standard strictly quasiconcave utility functions. For example, Champsaur–Cornet (Ref. 9) and later Bottazzi (Ref. 10) examine exchange processes where the agents have strictly quasiconcave utility functions. At every instant of time, they define a tangent economy where the agents exchange small amounts with respect to a linear short-term utility function which changes over time. This linear utility function is defined as the normal vector to the indifference surface at the present endowment point. In this model, the linear utility functions are of course no longer fixed parameters but depend on the time. Following another direction, as in Mertens (Ref. 6), the linear exchange model may also be used to study markets with limit price orders. Then a linear utility function can be interpreted as an exchange rate or a personal price at which one agrees to exchange one good for another, and the initial endowment in the linear exchange economy corresponds to the quantities that one is willing to sell on the market.

The purpose of this paper is to examine for linear exchange economies the type of questions which have been studied in the standard case in many papers since the pioneering work of Debreu (Ref. 11); see for example Refs. 12–16. Indeed, we study how the equilibrium prices and equilibrium allocations vary with respect to the initial endowments and utility vectors and we characterize also the multiplicity of equilibria.

The study of equilibrium prices is equivalent to studying the equilibrium manifold, which is the graph of the mapping associating the equilibrium price vectors to the initial endowments and utility vectors. Since the demand associated with a linear utility function is not single-valued, but a correspondence which is only upper semicontinuous for strictly positive prices, we cannot use the classical tools of differential geometry in this work.

Another way to attack this question is to use the tools of sensitivity analysis for optimization problems. Indeed, in Cornet (Ref. 5), it is proved that an equilibrium is a solution of a convex optimization program in which the initial endowments and utility vectors appear as parameters. However, the objective function of this program is defined through the value of another program with a multiplicity of solutions. The sensitivity analysis of this last type of problems is nowadays a topic of active research.

In our paper, we will be interested in two issues, the uniqueness of the Walras equilibrium and the behavior of the equilibrium allocation and price correspondence. Further properties, such as the differentiability of the price function (when a normalization for the price is chosen and the price is unique up to positive scale multiplication) and the property of gross substitution are studied in Bonnisseau–Florig–Jofré (Ref. 8). The uniqueness of the utility level at Walrasian equilibrium and the uniqueness of the Walrasian equilibrium price under a survival assumption are well known (Refs. 3, 5). The upper semicontinuity of the equilibrium price and allocation correspondence has been studied already under more restrictive assumptions in Champsaur–Cornet (Ref. 9).

The outline of the paper is the following. In Section 2, we establish the model. In Section 3, we study some properties, not always very well-known, of linear exchange economies such as the upper semicontinuity of the equilibrium allocation correspondence, the closedness of the equilibrium price correspondence on the space of the utility vectors and the initial endowments. The convex valuedness of both correspondences will also be established. Comparable results can be found in Cornet (Ref. 5), Mertens (Ref. 6), Champsaur–Cornet (Ref. 9). Mainly, Cheng (Ref. 17) attempted to formulate a necessary and sufficient condition for the uniqueness of the equilibrium price (up to positive scale multiplication). In fact, his condition is identical to the Gale sufficient condition (Ref. 3) for the uniqueness of the equilibrium price. In this sense, we give in Section 4 a neccessary and sufficient condition for the uniqueness up to positive scale multiplication of the equilibrium price, thereby generalizing these previous works. We characterize also the economies for which the equilibrium allocation is unique. Later, in Section 5, we attack the more difficult problem concerning the lower semicontinuity of the equilibrium allocation correspondence. Indeed, unlike the standard case, the equilibrium allocation correspondence is not only upper semicontinuous but as well lower semicontinuous when the utility vectors remain fixed. This result, jointly with the convexity of the values of this correspondence, gives the existence of equilibrium allocation selections which are continuous with respect to the initial endowments. Furthermore, we characterize the full measure set where the equilibrium allocation correspondence is single-valued and lower semicontinuous with respect to the initial endowments and utility vectors.

These results are deduced from the graph properties associated to the equilibrium prices and they will be used in the continuation of this paper (Ref. 8) in order to obtain some differentiability properties of the price and allocations.

2. Model

We consider a linear exchange economy with a finite set $L=\{1,\ldots,l\}$ of commodities and a finite set $I=\{1,\ldots,m\}$ of consumers. The consumption set of consumer i is R^l_+; his utility function u_i: $R^l_+\to R$ is defined by

$$u_i(x_i)=b_i\cdot x_i$$

for a given vector $b_i\in R^l_+$. His initial endowment is a vector ω_i in R^l_+. For each $(b,\omega)\in((R^l_+)^m)^2$, $\mathscr{L}$ (b,ω) denotes the linear exchange economy associated with the parameters b and ω. Throughout the paper, we will make the following assumptions:

(A1) $(\sum_{i=1}^m b_i, \sum_{i=1}^m \omega_i)\in R^l_{++}\times R^l_{++}$;

(A2) for every i, $b_i\neq 0$ and $\omega_i\neq 0$.

Condition (A1) simply means that every good is desired by at least some consumer and is owned by at least a consumer. Condition (A2) means that every consumer desires at least one good and owns at least one good. Now, we recall some standard definitions and known results. Their proofs can be found for example in Gale (Refs. 1–3) and Cornet (Ref. 5).

Definition 2.1.

(i) If $p\in R^l_+$ is a price vector, the demand of consumer i, denoted $d(b_i,p,p\cdot.\omega_i)$, is the set of solutions of the following maximization problem:

$$\max u_i(x_i)=b_i\cdot x_i,$$

$$p\cdot x_i\leq p\cdot\omega_i,$$

$$x_i\geq 0.$$

(ii) A Walras equilibrium of $\mathscr{L}(b, \omega)$ is an element $(x, p) \in (R^l_+)^m \times R^l_+$ such that:

(a) for every i, $x_i \in d(b_i, p, p \cdot \omega_i)$;

(b) $\sum_{i=1}^m x_i = \sum_{i=1}^m \omega_i$.

(iii) A proper subset I' of I is called self-sufficient in $\mathscr{L}(b, \omega)$ if, for all $h \in L$, $\sum_{i \in I'} b_{ih} > 0$ implies $\sum_{i \in I \setminus I'} \omega_{ih} = 0$.

(iv) A proper subset I' of I is called super self-sufficient in $\mathscr{L}(b, \omega)$ if it is self-sufficient and there exists $h \in L$ such that $\sum_{i \in I'} \omega_{ih} > 0$, but $\sum_{i \in I'} b_{ih} = 0$.

For every $(b, \omega) \in (R^l_+)^m \times (R^l_+)^m$, $X(b, \omega)$ is the set of Walrasian equilibrium allocations in $(R^l_+)^m$ and $P(b, \omega)$ is the set of Walrasian equilibrium price vectors in R^l_+. Note that, if the price p is not positive, then (A1) and (A2) imply that the demand of at least one consumer is empty. Thus, $P(b, \omega)$ is always included in R^l_{++}. For consumer i, the marginal rate of substitution between the commodities h and k, denoted $r(b_i, h, k)$, is b_{ih}/b_{ik}, where by convention $0/0 = 0$ and $b_{ih}/0 = +\infty$ if $b_{ih} > 0$. For each $p \in R^l_{++}$,

$$\delta(b_i, p) = \{h \in L \mid p_h \leq r(b_i, h, k) p_k, \forall k \in L\}.$$

$\delta(b_i, p)$ is the set of commodities that the consumer wishes to consume if the price vector is p, since the ratio between the marginal utility and the price is maximal for these commodities. More precisely, let ϵ^h be the vector of R^l whose coordinates are equal to 0 except the hth, which is equal to 1. Then, for all $p \in R^l_{++}$, $d(b_i, p, p \cdot \omega_i)$ is the convex hull of the points

$$((p \cdot \omega_i / p_h) \epsilon^h)_{h \in \delta(b_i, p)}. \tag{1}$$

In other words, $x_i \in d(b_i, p, p \cdot \omega_i)$ if and only if $p \cdot x_i = p \cdot \omega_i$ and $\operatorname{supp}(x_i) \subset \delta(b_i, p)$, where for $y \in R^n$,

$$\operatorname{supp}(y) = \{h \in \{1, \ldots, n\} \mid y_h \neq 0\}$$

is the support of y. We recall that each equilibrium allocation has the same utility level (Gale, Ref. 3); that is, for every $x, x' \in X(b, \omega)$, for every $i \in I$,

$$b_i \cdot x_i = b_i \cdot x'_i.$$

If (x, p) is an equilibrium of $\mathscr{L}(b, \omega)$, then for every i,

$$b_i \cdot x_i = v(b_i, p, p \cdot \omega_i),$$

where v is the indirect utility function, that is, the mapping from $R^l_+ \times R^l_{++} \times R$ defined by

$$v(b_i, p, w_i) = w_i \max\{b_{ih}/p_h \mid h \in L\}.$$

Usually in the literature, the indirect utility function depends on only the price and the income of the consumer. Since we want to consider the utility vectors as parameters, we extend the standard definition of indirect utility function by considering the vector b_i, as an argument of v.

In Cornet (Ref. 5), the existence of an equilibrium is proved by considering the following maximization problem:

$$\max\min\{b_i\cdot x_i - v(b_i, p, p\cdot\omega_i) \mid i\in I\},$$

$$\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} \omega_i,$$

$$x_i\in R^l_+, \qquad \text{for all } i\in I,$$

$$p\in R^l_{++}.$$

The author of Ref. 5 proves that this problem always has a solution and a solution is an equilibrium of the economy. The converse implication is obvious. Consequently, the study of the equilibria in a linear exchange economy can be seen as an analysis of sensitivity for a maximization problem depending on the parameters ω and b. Nevertheless, the existing results in this domain are not very useful, since the assumptions usually made are not satisfied in our framework. Indeed, neither the equilibrium price nor the equilibrium allocation need to be unique.

A subset of the set of traders is self-sufficient if they own the whole quantity of goods they are interested in and it is called super self-sufficient if they own as well a positive amount of some good which nobody in their subgroup is interested in. An economy such that no such super self-sufficient subset exists is called irreducible. Let us now introduce some notations which are used throughout the paper. We will denote by $\mathscr{W}\subset (R^l_+)^m\times(R^l_+)^m$ the set of pairs (b,ω) such that $\mathscr{L}(b,\omega)$ is irreducible and such that Conditions (A1) and (A2) are satisfied. Under (A1), the nonexistence of a super self-sufficient set is a necessary and sufficient condition for the existence of a Walras equilibrium in linear exchange economies (Ref. 3).

We further denote by $\mathscr{V}$ the pairs $(b,\omega)\in\mathscr{W}$ such that $\mathscr{L}(b,\omega)$ has no proper self-sufficient subset, and we denote by $\mathscr{U}$ the pairs $(b,\omega)\in\mathscr{W}$ such that $\mathscr{L}(b,\omega)$ has a unique equilibrium price vector $p(b,\omega)$ up to positive scale multiplication.

In the following, we use extensively subsets of $I\times L$ which are directly linked with graphs. Let p be a price vector in R^l_{++} and let

$$G(b,p) = \{(i,h)\in I\times L \mid h\in\delta(b_i,p)\},$$

$$G^+(b,\omega) = \{(i,h)\in I\times L \mid \exists x\in X(b,\omega), x_{ih}>0\}.$$

From (1), one deduces that

$$G^{+}(b,\omega)\subset G(b,p(b,\omega)),\qquad \text{for all } p(b,\omega)\in P(b,\omega).$$

Furthermore, note that $G^{+}(b,\omega)$ [resp. $G(b,p)$] may be seen as a graph where the set of vertices is $I\times L$ and there exists an edge between the vertices i and h if and only if $(i,h)\in G^{+}(b,\omega)$ [resp. $(i,h)\in G(b,p)$].

3. Upper Semicontinuity of the Correspondences, X and P

The following proposition gathers the results of this section and we then give an example which shows that the correspondence P is not upper semicontinuous on $\mathscr{W}$.

Proposition 3.1.

(i) If $(b,\omega)\in\mathscr{W}$, then every pair (x,p) in $X(b,\omega)\times P(b,\omega)$ is a Walras equilibrium of $\mathscr{E}(b,\omega)$.

(ii) The correspondence $X\colon \mathscr{W}\to(R^{l}_{+})^{m}$ is upper semicontinuous and has nonempty, convex, and compact values.

(iii) The correspondence $P\colon \mathscr{W}\to R^{l}_{++}$ has a closed graph and nonempty and convex values; furthermore, if we choose a normalization, then it is upper semicontinuous on $\mathscr{V}$ and it reduces to a continuous mapping on $\mathscr{U}$.

Note that the convexity of $X(b,\omega)$ implies that there exists $x\in X(b,\omega)$ such that

$$x_{ih}>0,\qquad \text{for all } (i,h)\in G^{+}(b,\omega).$$

Proof of Proposition 3.1.

(i) Let $(b,\omega)\in\mathscr{W}$, and let $(x,p)\in X(b,\omega)\times P(b,\omega)$. By the uniqueness of the utility level at equilibrium,

$$b_i\cdot x_i=v(b_i,p,p\cdot\omega_i).$$

Hence, for every $i\in I$,

$$p\cdot x_i\geq p\cdot\omega_i.$$

Since

$$\sum_{i=1}^{m}x_i=\sum_{i=1}^{m}\omega_i,$$

we deduce that, for every $i \in I$,

$$p \cdot x_i = p \cdot \omega_i.$$

Therefore, (x, p) is a Walras equilibrium of $\mathscr{L}(b, \omega)$.

(ii) The nonemptiness of $X(b, \omega)$ comes from Gale (Ref. 3) and the convexity is a direct consequence of (i) together with the uniqueness of the utility level at equilibrium (Ref. 3). The compactness is obvious. In order to show that X is upper semicontinuous, it is sufficient to show that X is locally bounded and has a closed graph in $\mathscr{W} \times (R^l_+)^m$. For every $x \in X(b, \omega)$ and every i,

$$0 \leq x_i \leq \sum_{i=1}^{m} \omega_i,$$

and therefore X is locally bounded. Let (b^n, ω^n, x^n) be a sequence in the graph of X with (b^n, ω^n) converging to $(b, \omega) \in \mathscr{W}$ and x^n converging to $x \in (R^l_+)^m$. For every n, let $p^n \in P(b^n, \omega^n)$, and without loss of generality suppose that p^n converges to $p \in R^l_+ \setminus \{0\}$; indeed, it is always possible to normalize the prices into the unit simplex, which is compact and does not contain 0. We have to show that $x \in X(b, \omega)$. Clearly,

$$\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} \omega_i$$

and

$$p \cdot x_i = p \cdot \omega_i, \qquad \text{for every } i.$$

Thus, it remains only to show that, for all i,

$$x_i \in d(b_i, q, q \cdot \omega_i), \qquad \text{for some } q \in P(b, \omega).$$

We can now differentiate two cases.

Case 1. $p \in R^l_{++}$. Since, $p \in R^l_{++}$ implies $p \cdot \omega_i > 0$, for all i, one checks easily that $x_i \in d(b_i, p, p \cdot \omega_i)$ for all i. Therefore, p is an equilibrium price vector of $\mathscr{L}(b, \omega)$ and $(x, p) \in X(b, \omega) \times P(b, \omega)$. Note that this step proves also that the graph of P is closed relatively to R^l_{++}.

Case 2. $p \notin R^l_{++}$. We construct a price vector q such that (q, x) is an equilibrium. For this, we consider a partition of the set of commodities and of the set of consumers, which is built step-by-step. In the following, if S is a subset of L and x a vector of R^l, $x_{|S}$ is the restriction of x to the coordinates in S. Let

$$S_1 = \operatorname{supp}(p), \quad S_1' = L \setminus S_1, \quad I_1 = \{i \in I \mid p \cdot \omega_i > 0\}, \quad I_1' = I \setminus I_1.$$

One checks easily that, for all $i \in I_1$, $x_i \in d(b_i, p, p \cdot \omega_i)$, hence for all $h \in S'_1$, $b_{ih} = 0$. Note that this implies that x_i remains in the demand even if one changes the prices for the commodities in S'_1. For all $i \in I'_1$, $x_{i|S_1} = 0$; thus,

$$\sum_{i \in I_1} x_{i|S_1} = \sum_{i \in I} \omega_{i|S_1} \geq \sum_{i \in I_1} \omega_{i|S_1}.$$

Since

$$p_{|S_1} \cdot \sum_{i \in I_1} x_{i|S_1} = p_{|S_1} \cdot \sum_{i \in I_1} \omega_{i|S_1},$$

one deduces that

$$\sum_{i \in I_1} \omega_{i|S_1} = \sum_{i \in I} \omega_{i|S_1}.$$

Finally, since I_1 is not a super self-sufficient set, for all $i \in I_1$, $\omega_{i|S'_1} = 0$. For all $i \in I'_1$, since $\omega_i \neq 0$ and since $I_1 \cup \{i\}$ is not a super self-sufficient set, there exists $h \in S'_1$ such that $b_{ih} > 0$. We now prove that, for n large enough, $x^n_{i|S_1} = 0$. For all $k \in S_1$, b^n_{ik}/p^n_k is bounded from above, whereas b^n_{ih}/p^n_h tends to $+\infty$. Consequently, for n large enough, k does not belong to $\delta(b^n_i, p^n)$ which implies $x^n_{ik} = 0$.

Let q^n be the normalization of $p^n_{|S'_1}$ in the simplex of $R^{S'_1}$. Without loss of generality, we can assume that the sequence (q^n) converges to some p_1 in the simplex of $R^{S'_1}$. We define

$$S_2 = \mathrm{supp}(p_1), \quad S'_2 = S'_1 \setminus S_2, \quad I_2 = \{i \in I'_1 \,|\, p_1 \cdot \omega_{i|S'_1} > 0\}, \quad I'_2 = I'_1 \setminus I_2.$$

First, we prove that

$$x_{i|S_2} = 0, \qquad \text{for all } i \notin I_2.$$

Note that the definition of I_2 and the fact that $\mathrm{supp}(\omega_i) \subset S_1$ for $i \in I_1$ imply that

$$\omega_{i|S_2} = 0, \qquad \text{for all } i \notin I_2.$$

Thus,

$$\sum_{i \in I} \omega_{i|S_2} = \sum_{i \in I_2} \omega_{i|S_2}.$$

For all $i \in I_2$,

$$p^n \cdot x^n_i = p^n \cdot \omega^n_i,$$

and for n large enough, $x^n_{i|S_1} = 0$. Consequently,

$$q^n \cdot x^n_{i|S'_1} \geq q^n \cdot \omega^n_{i|S'_1}.$$

At the limit, one obtains

$$p_1 \cdot x_{i|S'_1} = p_{1|S_2} \cdot x_{i|S_2} \geq p_1 \cdot \omega_{i|S'_1} = p_{1|S_2} \cdot \omega_{i|S_2}.$$

But

$$\sum_{i \in I_2} x_{i|S_2} \leq \sum_{i \in I} \omega_{i|S_2} = \sum_{i \in I_2} \omega_{i|S_2}.$$

Consequently,

$$\sum_{i \in I_2} x_{i|S_2} = \sum_{i \in I} \omega_{i|S_2}.$$

Let us consider the economy obtained by considering only the consumers in I'_1 and the commodities in S'_1. We now prove that, in this economy, for all $i \in I_2$, $x_{i|S'_1}$ belongs to $d(b_{i|S'_1}, p_1, \omega_{i|S'_1})$. If it is not true, since $p_1 \cdot \omega_{i|S'_1} > 0$, there exists $\xi_i \in R^{S'_1}$ such that

$$b_{i|S'_1} \cdot \xi_i > b_{i|S'_1} \cdot x_{i|S'_1} \quad \text{and} \quad p_1 \cdot \xi_i < p_1 \cdot \omega_{i|S'_1}.$$

Let $\tilde{\xi}_i$ be the vector of R^l defined by

$$\tilde{\xi}_{i|S'_1} = \xi_i \quad \text{and} \quad \tilde{\xi}_{i|S_1} = 0.$$

For n large enough,

$$p^n \cdot \tilde{\xi}_i < p^n \cdot \omega_i \quad \text{and} \quad b_i^n \cdot \xi_i > b_i^n \cdot x_i^n,$$

since $x^n_{i|S_1} = 0$. This contradicts the maximality of x^n.

The maximality of $x_{i|S'_1}$ implies that

$$b_{ih} = 0, \qquad \text{for all } (i, h) \in I_2 \times S'_2.$$

Consequently, for all $i \in I_2$, there exists $h \in S_2$ such that $b_{ih} > 0$. This implies that, for all $\eta > 0$ small enough, x_i belongs to $d(b_i, p_1^\eta, \omega_i)$, where p_1^η is defined by

$$p^\eta_{1|S_1} = p_{|S_1} \quad \text{and} \quad p^\eta_{1|S'_1} = \eta p_1.$$

This remains true if we change the prices of the commodities in S'_2.

By doing the same arguments as above, we prove that, for all $i \in I'_2$, there exists $h \in S'_2$ such that $b_{ih} > 0$ and, for n large enough $x^n_{i|S_2} = 0$. In a finite number of steps, one obtains a partition $(S_k)_{k=1,\ldots,r}$ of L, a partition $(I_k)_{k=1,\ldots,r}$ of I, and a collection of price vectors $(p, p_1, \ldots, p_{r-1})$. From the above construction, there exist positive real numbers $(\eta_\rho)_{\rho=2,\ldots,r}$ such that (q, x) is an equilibrium of $\mathscr{L}(b, \omega)$, where q is defined by $q_{|S_1} = p_{|S_1}$ and, for all $\rho = 2, \ldots, r$, $q_{|S_\rho} = \eta_\rho p_{\rho-1|S_\rho}$. This ends the proof of (ii).

(iii) The closedness of P has been proved previously in the first case of the proof of (ii) and the nonemptiness comes from Gale (Ref. 3). We now prove that, for $(b, \omega) \in \mathscr{W}$, $P(b, \omega)$ is convex. Let $x \in X(b, \omega)$, $(p, p') \in (P(b, \omega))^2$, and for any $\lambda \in]0, 1[$, let

$$p^\lambda = \lambda p + (1 - \lambda) p'.$$

By (ii), for every $i \in I$,

$$p^\lambda \cdot x_i = p^\lambda \cdot \omega_i$$

and

$$G^+(b, \omega) \subset G(b, p) \cap G(b, p') \neq \emptyset.$$

Let $(i, h) \in G(b, p) \cap G(b, p')$; then, for every $h' \in L$,

$$b_{ih}/p_h \geq b_{ih'}/p_{h'} \quad \text{and} \quad b_{ih}/p'_h \geq b_{ih'}/p'_{h'}.$$

Thus,

$$b_{ih}/p^\lambda_h \geq b_{ih'}/p^\lambda_{h'},$$

and hence,

$$G^+(b, \omega) \subset G(b, p) \cap G(b, p') \subset G(b, p_\lambda).$$

Consequently,

$$x_i \in d(b_i, p^\lambda, p^\lambda \cdot \omega_i),$$

and as

$$\sum_{i \in I} x_i = \sum_{i \in I} \omega_i,$$

we can conclude that (x, p^λ) is an equilibrium, hence $P(b, \omega)$ is convex.

We now prove that, if we choose a normalization, then the correspondence $P: \mathscr{V} \to R^l_{++}$ is upper semicontinuous and reduces to a continuous mapping on $\mathscr{U}$. It is sufficient to check that the graph of $P: \mathscr{V} \to R^l_{++}$ is closed in R^l_+ if the prices are normalized into the unit simplex of R^l_+. Let (p^n, b^n, ω^n) be a sequence of the graph of P on $\mathscr{V}$ converging to $(p, b, \omega) \in (R^l_+ \setminus \{0\}) \times \mathscr{V}$ for this normalization. Clearly, if $p \in R^l_{++}$, then by the closedness of the graph of P in $\mathscr{W} \times R^l_{++}$, $p \in P(b, \omega)$. Otherwise, if $p \notin R^l_{++}$, then it is not an equilibrium price vector. Let

$$S_1 = \operatorname{supp}(p) \quad \text{and} \quad I_1 = \{i \in I \mid p \cdot \omega_i > 0\}.$$

I_1 is nonempty, since $\sum_{i \in I} \omega_i \gg 0$, and $I_1 \neq I$, since $p \notin P(b, \omega)$. For all $i \in I_1$,

$$\operatorname{supp}(b_i) \subset S_1 \quad \text{and} \quad \sum_{i \in I_1} \omega_{i|S_1} = \sum_{i \in I} \omega_{i|S_1}.$$

Consequently, I_1 is self-sufficient. This contradicts that $(b, \omega) \in \mathscr{V}$. We prove below in Corollary 4.1 that $\mathscr{U} \subset \mathscr{V}$, which shows the continuity of P on $\mathscr{U}$. $\square$

We now give an example where the correspondence $P: \mathscr{W} \to R^l_{++}$ is not upper semicontinuous. Consider an economy with three agents and three commodities. Let

$$b_1 = (1, 0, 0), \qquad b_2 = (0, 1, 0), \qquad b_3 = (0, 1, 1);$$

the initial endowments are

$$\omega_1^t = (1, t, 0), \qquad \omega_2^t = (1, 1, 0), \qquad \omega_3^t = (0, 0, 1).$$

The consumers 1 and 2 form a self-sufficient subset. Choosing the second good as a numéraire, for all $t > 0$,

$$P(b, \omega^t) = \{(t, 1, s) \mid s \in]0, 1]\}.$$

Let

$$O = \{x \in R^3_{++} \mid x_3 > 1 - x_1\}.$$

Clearly, this defines an open neighborhood of $P(b, \omega^1)$, but for every integer $n > 0$,

$$(1 - 2/n, 1, 1/n) \in P(b, \omega^{(1-2/n)}) \quad \text{and} \quad (1 - 2/n, 1, 1/n) \notin O.$$

Thus, P is not upper semicontinuous at (b, ω^1).

4. Uniqueness of the Equilibrium Price and Equilibrium Allocation

The uniqueness of the equilibrium price vector on $\mathscr{U}$ does not imply that $X(b, \omega)$ is reduced to a singleton. Now, we characterize the cases where $P(b, \omega)$ and $X(b, \omega)$ are singletons.

For a given economy $\mathscr{E}(b, \omega)$, an equilibrium price vector $p \in P(b, \omega)$ is said to decompose the economy if there exists a partition (I_1, I_2) of the set of consumers I such that:

(a) I_1 and I_2 are both nonempty and, for all $h \in L$, $\sum_{i \in I_1} \omega_{ih} > 0$ if and only if $\sum_{i \in I_2} \omega_{ih} = 0$ for all $i \in I_2$;

(b) for $k = 1, 2, p$ is an equilibrium price vector for the economy obtained by restricting the economy $\mathscr{E}(b, \omega)$ to the set of consumers I_k.

Gale (Ref. 3) proved that, if an equilibrium price vector does not decompose the economy, then every other equilibrium price vector is proportional to it. In particular, this implies that a sufficient condition for the uniqueness of the equilibrium price vector is that one consumer i satisfies the strong survival assumption $\omega_i \in R^l_{++}$. In Ref. 17, the author asserts that a given economy has a unique equilibrium price up to constant scale multiplication if and only if no equilibrium price decomposes the economy. By Gale (Ref. 3), this condition is sufficient for the uniqueness, but the following counterexample shows that it is not necessary. Consider an economy with two consumers and two commodities. The utility vectors are

$$b_1 = (1, 1), \qquad b_2 = (1, 1),$$

and the initial endowments are

$$\omega_1 = (1, 0), \qquad \omega_2 = (0, 1).$$

The unique equilibrium price vector (up to positive scale multiplication) of this economy is

$$p = (1, 1),$$

with

$$X(b, \omega) = \{(1-\lambda, \lambda), (\lambda, 1-\lambda) \mid \lambda \in [0, 1]\},$$

and in particular,

$$(\omega_1, \omega_2) \in X(b, \omega).$$

Clearly, the equilibrium price vector $p = (1, 1)$ decomposes the economy and

$$x_1 = \omega_1, \qquad x_2 = \omega_2$$

are the respective equilibrium allocations. Thus the nondecomposability is not necessary for the uniqueness of the equilibrium price.

In Gale's sufficient condition (Ref. 3), if we replace $G^+(b, \omega)$ (he considers the goods consumed at a given equilibrium price vector) by $G(b, p(b, \omega))$, the resulting condition is necessary and sufficient for uniqueness as following proposition shows.

Proposition 4.1. Let $(b, \omega) \in \mathscr{W}$ and $p \in P(b, \omega)$. Then, $(b, \omega) \in \mathscr{U}$ if and only if the economy $\mathscr{L}(c, \omega)$, defined by

$$c_{ih} = \begin{cases} b_{ih}, & \text{if } (i, h) \in G(b, p), \\ 0, & \text{otherwise,} \end{cases}$$

has no proper self-sufficient subset.

Proof of Proposition 4.1. First, we prove the "if part". Let $(x, p) \in X(b, \omega) \times P(b, \omega)$. Note that $(x, p) \in X(c, \omega) \times P(c, \omega)$ and $(c, \omega) \in \mathscr{W}$. Suppose that (I_1, I_2) is a partition of I and that I_1 is self sufficient in the economy $\mathscr{S}(c, \omega)$. Note that

$$V_1 = \{h \in L \mid \exists i \in I_1, c_{ih} > 0\}$$

and

$$W_j = \left\{h \in L \mid \sum_{i \in I_j}, \omega_{ih} > 0\right\}, \qquad \text{for } j = 1, 2.$$

As I_1 is self sufficient, $V_1 \subset W_1$; and since it is not super self-sufficient, $W_1 \subset V_1$. Consequently, $V_1 = W_1$. Since I_1 is self sufficient, $W_2 \subset L \setminus V_1$.

It is not difficult to check that, for every $i \in I_1$,

$$x_{i|L \setminus V_1} = \omega_{i|L \setminus V_1} = 0.$$

As for every $i \in I_2$, $\omega_{i|V_1} = 0$, at equilibrium we must also have $x_{i|V_1} = 0$.

For $\lambda > 0$, let

$$\pi(\lambda) = \lambda \pi_1 + \pi_2,$$

where π_1 [resp. π_2] is the canonic embedding of $p_{|V_1}$ [resp. $p_{|L \setminus V_1}$] into R^l. We will show that there exists $\bar{\lambda} > 1$, such that $(x, \pi(\bar{\lambda})) \in X(b, \omega) \times P(b, \omega)$, leading to a contradiction of $(b, \omega) \in \mathscr{U}$.

For every $i \in I_1$, $\delta(b_i, p) \subset V_1$, and thus there exists $\bar{\lambda} > 1$ close enough to 1 such that

$$\delta(b_i, \pi(\bar{\lambda})) = \delta(b_i, p).$$

Now, for $i \in I_2$ and every $\lambda > 1$, one checks easily that

$$\delta(b_i, \pi(\lambda)) = \delta(b_i, p) \cap (L \setminus V_1).$$

From this, we can deduce that, for every $i \in I$,

$$\operatorname{supp}(x_i) \subset \delta(b_i, \pi(\bar{\lambda})).$$

For every $i \in I$,

$$\pi(\bar{\lambda}) \cdot x_i = \pi(\bar{\lambda}) \cdot \omega_i.$$

Therefore, $\pi(\bar{\lambda})$ is in $P(b, \omega)$ and thus $(b, \omega) \notin \mathscr{U}$.

We will now prove the "only if part". Suppose that

$$(p, q) \in (P(b, \omega)^2).$$

We define the vector $r \in R^l_{++}$ by

$$r_h = \sqrt{p_h q_h}, \qquad \text{for every } h \text{ in } L.$$

By Cornet (Ref. 5), $r \in P(b, \omega)$ and, for every i, there exists h_i such that $(i, h_i) \in G^+(b, \omega)$.

By the uniqueness of the utility level at equilibrium (Ref. 3), for every $i \in I$,

$$(1/2)v(b_i, p, p \cdot \omega_i) + (1/2)v(b_i, q, q \cdot \omega_i) = v(b_i, r, r \cdot \omega_i).$$

Hence, one has for all i,

$$(1/2)(p \cdot \omega_i(b_{ih_i}/p_{h_i}) + q \cdot \omega_i(b_{ih_i}/q_{h_i})) = r \cdot \omega_i(b_{ih_i}/r_{h_i}).$$

The arithmetic mean is greater than or equal to the geometric mean. The equality holds only when the two numbers are equal. Thus, for every i in I and every h in L,

$$\begin{aligned}(1/2)p_h \omega_{ih}/p_{h_i} + (1/2)q_h \omega_{ih}/q_{h_i} &\geq \sqrt{p_h \omega_{ih}/p_{h_i}}\,\sqrt{q_h \omega_{ih}/q_{h_i}} \\ &= r_h \omega_{ih}/r_{h_i}.\end{aligned}$$

The above equality and inequalities imply that, for every i in I and every h,

$$p_h \omega_{ih}/p_{h_i} = q_h \omega_{ih}/q_{h_i}.$$

Hence, for every $i \in I$ and every $h \in \mathrm{supp}(\omega_i)$,

$$p_h/q_h = p_{h_i}/q_{h_i};$$

thus, p and q are collinear on $\mathrm{supp}(\omega_i) \cup \{h_i\}$.

Define graph $\mathscr{X}$ as the graph with the set of vertices L and with an edge between two vertices (g, h) if, for some i, $(g, h) \subset \mathrm{supp}(\omega_i)$. Denote by $Z_1, \ldots, Z_k$ the set of vertices of its connected components. This is a partition of L and it induces a partition $(I_j)_{j=1}^k$ of the consumers where

$$I_j = \{i \in I \mid \mathrm{supp}(\omega_i) \subset Z_j\}.$$

It is now straightforward to prove that, for every $j \in \{1, \ldots, k\}$, p and q are collinear on $Z_j \cup (\bigcup_{i \in I_j} \{h_i\})$.

Therefore, for $j = 1, \ldots, k$, there exists $t_j > 0$ such that

$$q_h = t_j p_h, \qquad \text{for all } h \in Z_j.$$

To end the proof, it suffices to prove that

$$t_1 = t_2 = \cdots = t_k.$$

Actually, we prove that

$$t_1 = \min\{t_j \mid j = 1, \ldots, k\}.$$

Since the proof can be done symmetrically for every t_j, this implies that they are all equal.

Since I_1 is not self-sufficient in the economy $\mathscr{E}(c, \omega)$, there exists $(i, h_2) \in I_1 \times (L \setminus Z_1)$ such that $h_2 \in \delta(b_i, p)$. Say, $h_2 \in Z_2$. By the uniqueness of the utility level at equilibrium,

$$v(b_i, p, p \cdot \omega_i) = p \cdot \omega_i(b_{ih_2}/p_{h_2}) \geq q \cdot \omega_i(b_{ih_2}/q_{h_2})$$
$$= t_1 p_{|Z_1} \cdot \omega_{i|Z_1}(b_{ih_2}/t_2 p_{h_2}).$$

Therefore, $t_1 \leq t_2$. Applying this argument to $\bigcup_{j=1}^{r} I_r$ subsequently for $r = 2, \ldots, k$, we deduce that

$$t_1 = \min\{t_j | j = 1, \ldots, k\}. \qquad \square$$

We cannot use directly the vectors b in the above proposition. Indeed, the following example indicates that we require to introduce the vector c. Consider an economy with two consumers and two commodities. The utility vectors are

$$b_1 = (2, 1), \qquad b_2 = (1, 2)$$

and the initial endowments are

$$\omega_1 = (1, 0), \qquad \omega_2 = (0, 1).$$

The equilibrium price of this economy is not unique, but $\mathscr{E}(b, \omega)$ has no proper self-sufficient subset. Eaves (Ref. 4) proved that it is always possible to compute with the Lembke algorithm a Walrasian equilibrium in a finite number of steps. Thus, assuming an equilibrium price as given is not such an expensive requirement, since it is possible to compute one in finite time. Then, given an economy $\mathscr{E}(b, \omega)$, it is possible to check in a finite number of steps whether the equilibrium price is unique up to positive scale multipliction or not.

Corollary 4.1. We have $\mathscr{U} \subset \mathscr{V} \subset \mathscr{W}$.

By definition, $\mathscr{U}$ and $\mathscr{V}$ are included in $\mathscr{W}$. The first inclusion is an immediate consequence of Proposition 4.1. Indeed, for (b, ω) to be in $\mathscr{U}$, we need (c, ω) to be in $\mathscr{V}$, and this is not possible if $(b, \omega) \notin \mathscr{V}$, since a self-sufficient subset of $\mathscr{E}(b, \omega)$ is also a self-sufficient subset of $\mathscr{E}(c, \omega)$.

The following proposition gives a necessary and sufficient condition for the uniqueness of the equilibrium allocation. In order to check this criterion, it is necessary to know the graph $G^+(b, \omega)$.

Proposition 4.2. Let (b, ω) in $\mathscr{W}$. $X(b, \omega)$ is a singleton if and only if the graph $G^+(b, \omega)$ has no (nondegenerate) cycle; this is, there does not exist a finite family $((i_1, h_1), \ldots, (i_n, h_n))$ of two-by-two different elements of $I \times L$ such that (i_ν, h_ν) and $(i_\nu, h_{\nu+1})$ belong to $G^+(b, \omega)$ where $h_{n+1} = h_1$.

The proof of this proposition uses the following lemma, which will be useful also in the proof of other results. Its proof is given in the Appendix (Section 6).

Lemma 4.1. Let (b, ω) in $\mathscr{W}$, and let x and $\bar{x}$ be two different elements of $X(b, \omega)$. Then, there exists a finite family $((i_1, h_1), \ldots, (i_n, h_n))$ of two-by-two different elements of $I \times L$ such that:

(i) for every $\nu = 1, \ldots, n$, $x_{i_\nu h_\nu} < \bar{x}_{i_\nu h_\nu}$, $\bar{x}_{i_\nu h_{\nu+1}} < x_{i_\nu h_{\nu+1}}$; hence, (i_ν, h_ν) and $(i_\nu, h_{\nu+1})$ belong to $G^+(b, \omega)$, where $h_{n+1} = h_1$;
(ii) $1 = \prod_{\nu=1}^n r(b_{i_\nu}, h_\nu, h_{\nu+1})$.

We remark that, for all $\nu \in \{1, \ldots, n\}$, $r(b_{i_\nu}, h_\nu, h_{\nu+1}) \in]0, \infty[$.

Proof of Proposition 4.2. The "only if part" is a direct consequence of the previous lemma. Let us now prove the converse implication by contraposition. If $X(b, \omega)$ is a singleton and if there exists a finite family $((i_1, h_1), \ldots, (i_n, h_n))$ of two-by-two different elements of $I \times L$ such that, for all ν, (i_ν, h_ν) and $(i_\nu, h_{\nu+1})$ belong to $G^+(b, \omega)$, where $h_{n+1} = h_1$, let x be the unique element of $X(b, \omega)$. From the definition of $G^+(b, \omega)$, $x_{ih} > 0$ for all $(i, h) \in G^+(b, \omega)$.

We exhibit an equilibrium allocation $\tilde{x}$ of $\mathscr{L}(b, \omega)$ which is deduced from the allocation x by modifying only the components $(x_{i_\nu h_\nu}, x_{i_\nu h_{\nu+1}})_{\nu=1,\ldots,n}$. Fix $p(b, \omega) \in P(b, \omega)$, and let $\alpha > 0$ such that

$$\alpha < \min\{p_{h_{\nu+1}}(b, \omega) x_{i_\nu h_{\upsilon+1}} \mid \nu = 1, \ldots, n\}.$$

Such α exists, since

$$p_{h_{\nu+1}}(b, \omega) x_{i_\nu h_{\nu+1}} > 0, \qquad \text{for all } \nu = 1, \ldots, n.$$

Let

$$\tilde{x}_{i_\nu h_\nu} = x_{i_\nu h_\nu} + \alpha / p_{h_\nu}(b, \omega) \quad \text{and} \quad \tilde{x}_{i_\nu h_{\nu+1}} = x_{i_\nu h_{\nu+1}} - \alpha / p_{h_{\nu+1}}(b, \omega).$$

Since (i_ν, h_ν) and $(i_\nu, h_{\nu+1})$ belong to $G^+(b, \omega)$, one checks easily that $\tilde{x}$ is an equilibrium allocation of $\mathscr{L}(b, \omega)$, which contradicts the fact that $X(b, \omega)$ is a singleton since $\tilde{x} \neq x$. □

The next proposition shows that it is possible to compute $G^+(b, \omega)$ in a finite number of steps. For this, it is sufficient to compute a Walras equilibrium by applying Eaves' algorithm (Ref. 4) and then the algorithm proposed here.

Proposition 4.3. Let (x, p) be a Walras equilibrium of $\mathscr{S}(b, \omega)$. Then, the following finite algorithm computes some $\xi \in X(b, \omega)$ such that $(i, h) \in G^{+}(b, \omega)$ if and only if $(i, h) \in \operatorname{supp} \xi$:

Step 1. Set $x^0 = x$ and set $r = 0$.

Step 2. Compute an element out of the set $\mathscr{F}^r$, which we define to be the set of families $((i_1, h_1), \ldots, (i_n, h_n))$ of two-by-two different elements of $I \times L$ such that, for all $j \in \{1, \ldots, n\}$, $x^r_{i_j h_{j-1}} > 0$ with $h_0 = h_n$, for all $j \in \{1, \ldots, n\}$, $(i_j, h_j) \in G(b, p)$, and $x^r_{i_1 h_1} = 0$. If $\mathscr{F}^r = \varnothing$, then go to Step 4; otherwise, go to Step 3.

Step 3. Choose $((i_1, h_1), \ldots, (i_n, h_n)) \in \mathscr{F}^r$. Let

$$\tau^r = \min_{j \in \{1, \ldots, n\}} p_{h_{j-1}} x^r_{i_j h_{j-1}},$$

with $h_0 = h_n$. Let $t^r \in (R^l)^m$ be defined as follows: for $(i, h) \notin ((i_1, h_1), \ldots, (i_n, h_n))$, $t^r_{ih} = 0$; for all $j \in \{1, \ldots, n\}$, $t^r_{i_j h_j} = \tau^r / (2 p_{h_j})$ and $t^r_{i_j h_{j-1}} = -\tau^r / (2 p_{h_{j-1}})$. Set

$$x^{r+1} = x^r + t^r;$$

set the counter to $r + 1$, and go back to Step 2.

Step 4. Set $\xi = x^r$.

Here, we did not specify how to compute $\mathscr{F}^r$ or to check that it is empty. However, note that it is possible to compute $\mathscr{F}^r$ in a finite number of steps. Indeed, $\mathscr{F}^r$ is a subset out of the set of all cycles of the finite graph $G(b, p)$ which pass through any vertex at most once, so a subset out of a finite set. Thus, one may compute all such cycles with a finite algorithm (Ref. 18). Then, for each potential element of $\mathscr{F}^r$, it takes again only a finite number of computations in order to check whether it is in $\mathscr{F}^r$ or not.

In order to prove the proposition, we use the following lemma which we prove in the Appendix (Section 6).

Lemma 4.2. Let (x, p) be a Walras equilibrium of $\mathscr{S}(b, \omega)$. Then, the following assertions are equivalent:

(i) for all $(i, h) \in G^{+}(b, \omega)$, $(i, h) \in \operatorname{supp} x$;

(ii) there exists no finite family $((i_1, h_1), \ldots, (i_n, h_n))$ of two-by-two different elements of $I \times L$ such that, for all $j \in \{1, \ldots, n\}$, $x_{i_j h_{j-1}} > 0$ with $h_0 = h_n$, for all $j \in \{1, \ldots, n\}$, $(i_j, h_j) \in G(b, p)$, and for at least one $j \in \{1, \ldots, n\}$, $x_{i_j h_j} = 0$.

Proof of Proposition 4.3. One may check that, for all r, for all $i \in I$,

$$p \cdot t^r_i = 0, \qquad x^r_i + t^r_i \geq 0, \qquad x^r_{ih} + t^r_{ih} > 0,$$

only if $(i, h) \in G(b, p)$ and $\sum_{i \in I} t_i^r = 0$. Therefore, $(x^r + t^r, p)$ is a Walras equilibrium of $\mathscr{E}(b, \omega)$. Moreover, for all $r \in N$ such that $\mathscr{F}^{r-1} \neq \emptyset$, $\operatorname{supp} x^{r-1}$ is a proper subset of $\operatorname{supp} x^r$. (Note that this implies that, if $\mathscr{F}^0 \neq \emptyset$ [i.e., if assertion (ii) in Lemma 4.2 does not hold], there exists $(i, h) \in \operatorname{supp} x^1 \setminus \operatorname{supp} x^0$, thereby contradicting assertion (i) in Lemma 4.2; hence assertion (i) of Lemma 4.2 implies assertion (ii) of Lemma 4.2.

This method computes iteratively Walras equilibria (x^r, p). The process stops with some $\xi \in X(b, \omega)$ after a finite number of iterations (less than ml) as soon as $\mathscr{F}^r = \emptyset$. By Lemma 4.2, $\xi \in X(b, \omega)$ such that $(i, h) \in G^+(b, \omega)$ if and only if $(i, h) \in \operatorname{supp} \xi$. □

The following proposition gives a sufficient condition on the utility functions in order to obtain a unique equilibrium allocation. This condition is faster to be checked than the necessary and sufficient condition. It is stated only in terms of the fundamentals of the economy.

Proposition 4.4. Let $\mathscr{B}$ be the set of elements $b \in (R_+^l)^m$ such that, for all finite families $((i_1, h_1), \ldots, (i_n, h_n))$ of two-by-two elements of $I \times L$,

$$\prod_{\nu=1}^{n} b_{i_\nu h_\nu} / b_{i_\nu h_{\nu+1}} \neq 1,$$

where $h_{n+1} = h_1$, $b_{i_\nu h_\nu}$ and $b_{i_\nu h_{\nu+1}}$ are different from zero for all $\nu \in \{1, \ldots, n\}$. Then, for all $(b, \omega) \in \mathscr{W} \cap (\mathscr{B} \times (R_+^l)^m)$, $X(b, \omega)$ is a singleton. Furthermore, the mapping $(b, \omega) \to X(b, \omega)$ is continuous on $\mathscr{W} \cap (\mathscr{B} \times (R_+^l)^m)$.

This result is a direct consequence of the previous Lemma 4.1, and the closedness of the graph of X. In particular, it implies that the set of economies with a unique equilibrium allocation contains a dense open subset of $(R_+^l)^m \times (R_+^l)^m$. As the following example shows, the fact that $X(b, \omega)$ is a singleton does not imply that $X(b, \omega')$ is a singleton for ω' close to ω. Consider an economy $\mathscr{E}(b, \omega)$ with three agents and three commodities. Let

$$b_1 = (2, 2, 1), \qquad b_2 = (1, 2, 2), \qquad b_3 = (2, 1, 2)$$

and

$$\omega_1 = (1, 1, 4), \qquad \omega_2 = (1, 1, 1), \qquad \omega_3 = (1, 1, 1).$$

The unique equilibrium of this economy is (x, p), with

$$p = (1, 1, 1), \qquad x_1 = (3, 3, 0), \qquad x_2 = (0, 0, 3), \qquad x_3 = (0, 0, 3).$$

Perturbing the initial endowment of the third consumer by

$$\omega_3^{\epsilon} = (1+\epsilon, 1, 1),$$

it is easy to check that, for every $\epsilon > 0$, (y, p), with

$$p = (1, 1, 1), \qquad y_1 = (3+t, 3-t, 0),$$
$$y_2 = (0, t, 3-t), \qquad y_3 = (\epsilon - t, 0, 3+t),$$

is a Walras equilibrium for every $t \in [0, \epsilon]$.

5. Lower Semicontinuity of X

We now study the lower semicontinuity of the correspondence X. The result differs depending on whether we consider fixed or variable utility functions. We will denote

$$\mathscr{W}_b = \{\omega \in (R_+^l)^m \,|\, (b, \omega) \in \mathscr{W}\}.$$

Proposition 5.1.

(i) For every $b \in (R_+^l)^m$, the correspondence $\omega \to X(b, \omega)$ is lower semicontinuous on $\mathscr{W}_b$.

(ii) The correspondence X from $\mathscr{W}$ to $(R_+^l)^m$ is lower semicontinuous at (b, ω) if and only if $X(b, \omega)$ is a singleton.

The proof of this proposition needs the following lemmata, which use Lemma 4.1 in their proofs. Their proofs are given in the Appendix. For all $\omega \in (R_+^l)^m$ and for all $r > 0$, let

$$B(\omega, r) = \left\{ \omega' \in (R_+^l)^m \,\middle|\, \sum_{i=1}^{m} \|\omega_i' - \omega_i\| < r \right\}.$$

Lemma 5.1. For every (b, ω) in $\mathscr{W}$, there exists $r > 0$ such that, for all $\omega' \in B(\omega, r) \cap \mathscr{W}_b$, $G^+(b, \omega) \subset G^+(b, \omega')$.

The conclusion of Lemma 5.1 does not hold if one perturbs the utility functions. Let us consider an economy with two goods and two consumers. Let

$$b_1 = b_2 = (1, 1) \quad \text{and} \quad \omega_1 = \omega_2 \in R_{++}^l.$$

Then, an equilibrium is

$$(\omega_1, \omega_2, p = (1, 1)).$$

Consequently,

$$G^+(b, \omega) = \{(1, 1), (1, 2), (2, 1), (2, 2)\}.$$

If we now consider the perturbed economy with the same parameters, except for the fact that

$$b_1^r = (1, 1 + r),$$

then $G^+(b^r, \omega)$ is included and different from $G^+(b, \omega)$ since b_1 and b_2 are not proportional. With the same example, one shows that S is not lower semicontinuous everywhere. Indeed, let

$$\omega_1 = \omega_2 = (1, 1).$$

Then,

$$X(b, \omega) = \{((t, 2 - t), (2 - t, t)) \mid t \in [0, 2]\}$$

and

$$X(b^r, \omega) = \{((0, 2), (2, 0))\}, \qquad \text{for } r > 0.$$

In the following lemma, we give a necessary and sufficient condition under which the conclusion of Lemma 5.1 holds even if one perturbs the utility functions.

Lemma 5.2. For every (b, ω) in $\mathscr{W}$, there exists $r > 0$ such that, for all $(b', \omega') \in (B(b, r) \times B(\omega, r)) \cap \mathscr{W}$, $G^+(b, \omega) \subset G^+(b', \omega')$ if and only if $X(b, \omega)$ is a singleton.

Proof of Proposition 5.1.

(i) Let $(b, \omega) \in \mathscr{W}$ such that the correspondence $X(b, \cdot)$ is not lower semicontinuous at this point. Then, there exists an open subset V of $(R_+^l)^m$ such that $X(b, \omega) \cap V \neq \varnothing$ and a sequence (ω^q) in $\mathscr{W}_b$ which converges to ω such that $X(b, \omega^q) \cap V = \varnothing$. Let (x^q) be a sequence such that $x^q \in X(b, \omega^q)$ for all q. Without loss of generality, we can assume that the sequence (x^q) converges to an element x and, since X has a closed graph, x belongs to $X(b, \omega)$. Let $\xi \in X(b, \omega) \cap V$. Since V is open, without loss of generality, we can assume that $\xi_{ih} > 0$ for all $(i, h) \in G^+(b, \omega)$. Let

$$y = \xi - x.$$

We finish this part of the proof by showing that, for q large enough,

$$\xi^q = x^q + y \in X(b, \omega^q),$$

which contradicts $X(b, \omega^q) \cap V = \varnothing$, since (ξ^q) converges to ξ.

First, we show that ξ^q is a nonnegative allocation for q large enough. If $y_{ih} < 0$, then $x_{ih} > 0$, which implies that $(i, h) \in G^+(b, \omega)$. Consequently, $\xi_{ih} > 0$; hence, for q large enough, ξ^q_{ih} which converges to ξ_{ih} is nonnegative. If $y_{ih} \geq 0$, then

$$\xi^q_{ih} = x^q_{ih} + y_{ih}$$

is obviously nonnegative. Note that $y_{ih} \neq 0$ only if $(i, h) \in G^+(b, \omega)$. From Lemma 5.1, for q large enough, $G^+(b, \omega) \subset G^+(b, \omega^q)$. Consequently,

$$\xi^q_{ih} = x^q_{ih} + y_{ih} > 0, \qquad \text{only if } (i, h) \in G^+(b, \omega^q).$$

To prove that $\xi^q \in X(b, \omega^q)$, it suffices to prove that

$$p(b, \omega^q) \cdot y_i = 0, \qquad \text{for all } i \text{ and some } p(b, \omega^q) \in P(b, \omega^q).$$

Since ξ and x belongs to $X(b, \omega)$, one has

$$p(b, \omega) \cdot y_i = 0, \qquad \text{for all } i \text{ and all } p(b, \omega) \in P(b, \omega).$$

Furthermore, since for q large enough, $G^+(b, \omega) \subset G^+(b, \omega^q)$, one has that, for all $((i, h), (i, h')) \in (G^+(b, \omega))^2$, for all $p(b, \omega) \in P(b, \omega)$, and all $p(b, \omega^q) \in P(b, \omega^q)$,

$$b_{ih}/b_{ih'} = p_{ih}(b, \omega^q)/p_{ih'}(b, \omega^q) = p_{ih}(b, \omega)/p_{ih'}(b, \omega).$$

Hence, for each i, for all $p(b, \omega) \in P(b, \omega)$, and all $p(b, \omega^q) \in P(b, \omega^q)$, the restrictions of the price vectors $p(b, \omega^q)$ and $p(b, \omega)$ to the commodities h such that $(i, h) \in G^+(b, \omega)$ are proportional. Consequently, since $y_{ih} \neq 0$ only if $(i, h) \in G^+(b, \omega)$,

$$p(b, \omega^q) \cdot y_i = 0, \qquad \text{for all } p(b, \omega^q) \in P(b, \omega^q).$$

This ends the first part of the proof.

(ii) Note that, if $X(b, \omega)$ is a singleton, the lower semicontinuity of X at (b, ω) is a direct consequence of the fact that X is upper semicontinuous. We now prove the converse implication. For this, first we prove that, if X is lower semicontinuous at (b, ω), then there exists $r > 0$ such that, for all $(b', \omega') \in (B(b, r) \times B(\omega, r)) \cap \mathscr{W}$,

$$G^+(b, \omega) \subset G^+(b', \omega').$$

Together with Lemma 5.2, this implies that $X(b, \omega)$ is a singleton.

Let us assume by contraposition that X is lower semicontinuous at (b, ω) and that, for all $r > 0$, there exists $(b^r, \omega^r) \in (B(b, r) \times B(\omega, r)) \cap \mathscr{W}$ and $(i^r, h^r) \in G^+(b, \omega)$ such that $(i^r, h^r) \notin G^+(b^r, \omega^r)$. Since $G^+(b, \omega)$ is finite, there exists a positive sequence (r^q) which converges to 0 such that the sequence (i^{r^q}, h^{r^q}) is constant equal to (i_1, h_1). For every converging sequence (x^{r^q}) such that $x^{r^q} \in X(b^{r^q}, \omega^{r^q})$, one has $x^{r^q}_{i_1 h_1} = 0$, since $(i_1, h_1) \notin G^+(b^{r^q}, \omega^{r^q})$.

Consequently, at the limit, $x_{i_1h_1}=0$. Since $(i_1,h_1)\in G^{+}(b,\omega)$, there exists an element $\xi\in X(b,\omega)$ such that $\xi_{i_1h_1}>0$. This proves that no sequence (x^{r^q}) such that $x^{r^q}\in X(b^{r^q},\omega^{r^q})$ converges to ξ, which contradicts the fact that X is lower semicontinuous at (b,ω). □

6. Appendix: Proofs of the Lemmata

Proof of Lemma 4.1.

(i) Since x is different from $\bar{x}$ and

$$\sum_{i=1}^{m} x_i=\sum_{i=1}^{m}\bar{x}_i=\sum_{i=1}^{m}\omega_i,$$

there exists $(i_1,h_1)\in I\times L$ such that $x_{i_1h_1}<\bar{x}_{i_1h_1}$. Since $x_{i_1h_1}<\bar{x}_{i_1h_1}$ and for all $p(b,\omega)\in P(b,\omega)$,

$$p(b,\omega)\cdot x=p(b,\omega)\cdot\bar{x},$$

there exists $h_2\in L$ such that $h_2\neq h_1$ and $\bar{x}_{i_1h_2}<x_{i_1h_2}$. Since

$$\sum_{i=1}^{m} x_{ih_2}=\sum_{i=1}^{m}\bar{x}_{ih_2}=\sum_{i=1}^{m}\omega_{ih_2},$$

there exists $i_2\in I$ such that $i_2\neq i_1$ and $0\leq x_{i_2h_2}<\bar{x}_{i_2h_2}$. With the same argument, we get a sequence $(i_\nu,h_\nu)_{\nu\geq 1}$ in $I\times L$ such that $x_{i_\nu h_\nu}<\bar{x}_{i_\nu h_\nu}$ and $\bar{x}_{i_\nu h_{\nu+1}}<x_{i_\nu h_{\nu+1}}$. Consequently, (i_ν,h_ν) and $(i_\nu,h_{\nu+1})$ are in $G^{+}(b,\omega)$.

Applying the same argument in the other direction, we obtain a sequence $(i_\nu,h_\nu)_{\nu\leq 1}$ in $I\times L$ such that $x_{i_\nu h_\nu}<\bar{x}_{i_\nu h_\nu}$ and $\bar{x}_{i_\nu h_{\nu+1}}<x_{i_\nu h_{\nu+1}}$; hence, (i_ν,h_ν) and $(i_\nu,h_{\nu+1})$ are in $G^{+}(b,\omega)$.

Since $I\times L$ is finite, there exist $\nu>1$ and $\nu'\leq 1$ such that $(i_\nu,h_\nu)=(i_{\nu'},h_{\nu\underline{\vee}})$. The family $\{(i_1,h_1),\ldots,(i_\nu,h_\nu),(i_{\nu'+1},h_{\nu'+1}),\ldots,(i_0,h_0)\}$ satisfies the conclusion of (i) if we choose the first ν and ν' such that $(i_\nu,h_\nu)=(i_{\nu'},h_{\nu'})$.

(ii) This equality is a direct consequence of the fact that, for every $\nu=1,\ldots,n$, (i_ν,h_ν) and $(i_\nu,h_{\nu+1})$ are in $G^{+}(b,\omega)$, where $h_{n+1}=h_1$; thus, for all $p(b,\omega)\in P(b,\omega)$,

$$b_{i_\nu h_\nu}/p_{h_\nu}(b,\omega)=b_{i_\nu h_{\nu+1}}/p_{h_{\nu+1}}(b,\omega),$$

where $h_{n+1}=h_1$. □

Proof of Lemma 4.2. The proof that assertion (i) implies assertion (ii) is contained in the proof of Proposition 4.3.

We prove by contraposition that assertion (ii) implies assertion (i). Suppose that there exists $(i,h)\in G^{+}(b,\omega)$ such that $x_{ih}=0$. Let $\xi\in X(b,\omega)$ with $\xi_{ih}>0$ and, as $X(b,\omega)$ is convex, we may assume that $\operatorname{supp} x$ is a proper subset of $\operatorname{supp}\xi$.

Let

$$t^{0}=\xi-x, \qquad \tau^{0}=\min_{(i,h)\in\operatorname{supp}t^{0}}|p_h t^{0}_{ih}|,$$

and let

$$(i_1,h_1)\in \operatorname*{argmin}_{(i,h)\in\operatorname{supp}t^{0}}|p_h t^{0}_{ih}|.$$

By the same argument as in the proof of the previous lemma, we may construct a finite family $((i_1,h_1),\dots,(i_n,h_n))$ of two-by-two different elements of $I\times L$ such that, for all $j\in\{1,\dots,n\}$, $t^{0}_{i_jh_j}t^{0}_{i_1h_1}>0$ and $t^{0}_{i_jh_j}t^{0}_{i_jh_{j-1}}<0$ with $h_0=h_n$. Let $\theta^{0}\in(R^{l})^{m}$ such that, for all $j\in\{1,\dots,n\}$,

$$\theta^{0}_{i_jh_j}=\tau^{0}t^{0}_{i_jh_j}/(p_{h_j}|t^{0}_{i_jh_j}|), \qquad \theta^{0}_{i_jh_{j-1}}=\tau^{0}t^{0}_{i_jh_{j-1}}/(p_{h_{j-1}}|t^{0}_{i_jh_{j-1}}|),$$

with $h_0=h_n$ and $\theta^{0}_{ij}=0$ otherwise. Now, one may check that $(x+\theta^{0},p)$ is a Walras equilibrium of $\mathscr{S}(b,\omega)$ and that $\operatorname{supp}t^{1}$, with $t^{1}=\xi-x-\theta^{0}$, is a proper subset of $\operatorname{supp}t^{0}$. Inductively, one may construct a finite family $(\theta^{0},\dots,\theta^{k})\in((R^{l})^{m})^{k+1}$ such that, for all $r\in\{1,\dots,k\}$, $\operatorname{supp}(\xi-x-\sum_{\rho=0}^{r}\theta^{\rho})$ is a proper subset of $\operatorname{supp}(\xi-x-\sum_{\rho=0}^{r-1}\theta^{\rho})$ and $(x+\sum_{\rho=0}^{r}\theta^{\rho},p)$ is a Walras equilibrium of $\mathscr{S}(b,\omega)$ and $x+\sum_{\rho=0}^{k}\theta^{\rho}=\xi$.

Let $r\in\{0,\dots,k\}$ be such that

$$\operatorname{supp}\left(x+\sum_{\rho=0}^{r-1}\theta^{\rho}\right)=\operatorname{supp}x$$

is a proper subset of $\operatorname{supp}(x+\sum_{\rho=0}^{r}\theta^{\rho})$. For all $(i,h)\in I\times L$, $t^{0}_{ih}\geq 0$ if and only if, for all $\rho\in\{0,\dots,k\}$, $\theta^{\rho}_{ih}\geq 0$, and $t^{0}_{ih}=\sum_{\rho=0}^{k}\theta^{\rho}_{ih}$, thus $(x+\theta^{r},p)$ is a Walras equilibrium of $\mathscr{S}(b,\omega)$. Moreover, $\operatorname{supp}x$ is a proper subset of $\operatorname{supp}(x+\theta^{r})$. The finite family $((i_1,h_1),(i_2,h_2),\dots,(i_n,h_n))$ of two-by-two different elements of $I\times L$ which we constructed in order to compute θ^{r} was chosen such that, for all $j\in\{1,\dots,n\}$, $x_{i_jh_{j-1}}>0$ with $h_0=h_n$, for all $j\in\{1,\dots,n\}$, $(i_j,h_j)\in G(b,p)$, and by the strict inclusion of the supports for at least one $j\in\{1,\dots,n\}$, $x_{i_jh_j}=0$. This contradicts assertion (ii). $\square$

Proof of Lemma 5.1. Let us assume that the conclusion of Lemma 5.1 is false. Then, there exists $(b,\omega)\in\mathscr{W}$ and a sequence $(\omega^{q},i^{q},h^{q})\in(\mathscr{W}_b)\times I\times L$ such that (ω^{q}) tends to ω and, for every q, $(i^{q},h^{q})\in G^{+}(b,\omega)$, $(i^{q},h^{q})\notin G^{+}(b,\omega^{q})$.

Since $I\times L$ is finite, we can assume without loss of generality that the sequence (i^q, h^q) is constant and equal to (i_1, h_1). For every q, let $x^q\in X(b, \omega^q)$. Since (ω^q) converges to ω, the sequence (x^q) is bounded; hence, we can assume without loss of generality that the sequence (x^q) converges to x. Since X has a closed graph, x belongs to $X(b, \omega)$. Furthermore, since $(i_1, h_1)\notin G^+(b, \omega^q)$, one has $x^q_{i_1}h_1 = 0$, hence $x_{i_1h_1} = 0$. From the definition of $G^+(b, \omega)$, there exists $\bar{x}\in X(b, \omega)$ such that $\bar{x}_{i_1h_1} > 0$.

From Lemma 4.1, there exists a finite family $(i_\nu, h_\nu)_{\nu=1,\ldots,n}$ of $I\times L$ such that:

(i) for every $\nu = 1, \ldots, n$, $x_{i_\nu h_\nu} < \bar{x}_{i_\nu h_\nu}$, $\bar{x}_{i_\nu h_{\nu+1}} < x_{i_\nu h_{\nu+1}}$; hence, (i_ν, h_ν) and $(i_\nu, h_{\nu+1})$ belong to $G^+(b, \omega)$ where $h_{n+1} = h_1$;

(ii) $1 = \prod_{\nu=1}^{n} r(b_{i_\nu}, h_\nu, h_{\nu+1})$.

For q large enough, for all $\nu = 1, \ldots, n$, one has $x^q_{i_\nu h_{\nu+1}} > 0$; hence, $(i_\nu, h_{\nu+1})\in G^+(b, \omega^q)$. We now show that, for all $p(b, \omega^q)\in P(b, \omega^q)$ and all ν, $(i_\nu, h_\nu)\in G(b, p(b, \omega^q))$. Indeed, if it is not true, then there exists $\bar{\nu}\in \{1, \ldots, n\}$ such that, for some $p(b, \omega^q)\in P(b, \omega^q)$,

$$b_{i_{\bar{\nu}}h_{\bar{\nu}}}/p_{h_{\bar{\nu}}}(b, \omega^q) < b_{i_{\bar{\nu}}h_{\bar{\nu}+1}}/p_{h_{\bar{\nu}+1}}(b, \omega^q).$$

Furthermore, for all $\nu = 1, \ldots, n$,

$$b_{i_\nu h_\nu}/p_{h_\nu}(b, \omega^q) \leq b_{i_\nu h_{\nu+1}}/p_{h_{\nu+1}}(b, \omega^q).$$

Hence, for all ν,

$$p_{h_\nu}(b, \omega^q) \geq r_{i_\nu}(b, h_\nu, h_{\nu+1})p_{h_{\nu+1}}(b, \omega^q),$$

with at least one strict inequality.

From (ii), one deduces that, for some $p(b, \omega^q)\in P(b, \omega^q)$,

$$p_{h_1}(b, \omega^q) > \prod_{\nu=1}^{n} r(b_{i_\nu}, h_\nu, h_{\nu+1})p_{h_1}(b, \omega^q) = p_{h_1}(b, \omega^q).$$

Hence, one obtains a contradiction.

We end the proof of Lemma 5.1 by showing that $(i_1, h_1)\in G^+(b, \omega^q)$, which leads to a contradiction. For this, we exhibit an equilibrium allocation $\tilde{x}^q$ of $\mathscr{L}(b, \omega^q)$ which is deduced from the allocation x^q by modifying only the components $(x^q_{i_\nu h_\nu}, x^q_{i_\nu h_{\nu+1}})_{\nu=1,\ldots,n}$. Fix $p(b, \omega^q)\in P(b, \omega^q)$, and let $\alpha > 0$ such that

$$\alpha < \min\{p_{h_{\nu+1}}(b, \omega^q)x^q_{i_\nu h_{\nu+1}} \mid \nu = 1, \ldots, n\}.$$

Let

$$\tilde{x}^q_{i_\nu h_\nu} = x^q_{i_\nu h_\nu} + \alpha/p_{h_\nu}(b, \omega^q),$$

$$\tilde{x}^q_{i_\nu h_{\nu+1}} = x^q_{i_\nu h_{\nu+1}} - \alpha/p_{h_{\nu+1}}(b, \omega^q).$$

For all $p(b,\omega^q)\in P(b,\omega^q)$, since (i_ν,h_ν) and $(i_\nu,h_{\nu+1})$ are in $G(b,p(b,\omega^q))$, one checks easily that $\tilde{x}$ in an equilibrium allocation of $\mathscr{L}(b,\omega^q)$. Furthermore, since $\tilde{x}_{i_1h_1}>0$, one deduces that $(i_1,h_1)\in G^+(b,\omega^q)$. □

Proof of Lemma 5.2. First, let us assume by contraposition that $X(b,\omega)$ is not a singleton. From Lemma 4.1, there exists a finite family $((i_1,h_1),\ldots,(i_n,h_n))$ of two-by-two different elements of $I\times L$ such that, for all $\nu=1,\ldots,n$, (i_ν,h_ν) and $(i_\nu,h_{\nu+1})$ belong to $G^+(b,\omega)$ where $h_{n+1}=h_1$.

Let $p\in P(b,\omega)$ and $x\in X(b,\omega)$. Without loss of generality, we assume that

$$p_{h_1}x_{i_1h_1}=\min\{p_{h_\nu}x_{i_\nu h_\nu}|\nu=1,\ldots,n\}.$$

Let $\rho\in]0,b_{i_1h_1}[$, and let b^ρ be deduced from b by modifying only b_{i_1} by $b^\rho_{i_1}=b_{i_1}-\rho\epsilon^{h_1}$. We exhibit an equilibrium $(\tilde{x},p)$ of $\mathscr{L}(b^\rho,\omega)$ which is deduced from the allocation x by modifying only the components $(x_{i_\nu h_\nu},x_{i_\nu h_{\nu+1}})_{\nu=1,\ldots,n}$. Let

$$\tilde{x}_{i_\nu h_\nu}=x_{i_\nu h_\nu}-p_{h_1}x_{i_1h_1}/p_{h_\nu}$$

and

$$\tilde{x}_{i_\nu h_{\nu+1}}=x_{i_\nu h_{\nu+1}}+p_{h_1}x_{i_1h_1}/p_{h_{\nu+1}}.$$

One checks easily that $(\tilde{x},p)$ is an equilibrium of $\mathscr{L}(b^\rho,\omega)$, since $\tilde{x}_{i_1h_1}=0$. Furthermore, (i_1,h_1) does not belong to $G(b^\rho,p)$; hence, (i_1,h_1) does not belong to $G^+(b^\rho,\omega)$. Since $(i_1,h_1)\in G^+(b,\omega)$, this implies that $G^+(b,\omega)$ is not a subset of $G^+(b^\rho,\omega)$ for $\rho>0$.

For the converse implication, we assume by contraposition that $X(b,\omega)$ is a singleton and, for all $r>0$, there exists $(b^r,\omega^r)\in(B(b,r)\times B(\omega,r))\cap\mathscr{W}$ and $(i^r,h^r)\in G^+(b,\omega)$ such that $(i^r,h^r)\notin G^+(b^r,\omega^r)$. Let $x^r\in X(b^r,\omega^r)$. Since X is upper semicontinuous, there exists a positive sequence (r^q) which converges to 0 such that the sequence (x^{r^q}) converges to x, the unique element of $X(b,\omega)$, and (i^{r^q},h^{r^q}) is constant and equal to (i_1,h_1). Since $(i_1,h_1)\notin G^+(b^{r^q},\omega^{r^q})$, one has $x^{r^q}_{i_1h_1}=0$, hence $x_{i_1h_1}=0$, which contradicts the fact that $(i_1,h_1)\in G^+(b,\omega)$. □

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