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Differentiability of Equilibria for Linear Exchange Economies[1]

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Abstract. The purpose of this paper is to study the differentiability properties of equilibrium prices and allocations in a linear exchange economy when the initial endowments and utility vectors vary. We characterize an open dense subset of full measure of the initial endowment and utility vector space on which the equilibrium price vector is a real analytic function, hence infinitely differentiable function. We provide an explicit formula to compute the equilibrium price and allocation around a point where it is known. We also show that the equilibrium price is a locally Lipschitzian mapping on the whole space. Finally, using the notion of the Clarke generalized gradient, we prove that linear exchange economies satisfy a property of gross substitution.

Key Words. General equilibrium, linear utility functions, equilibrium manifold, sensitivity analysis.

1. Introduction

The aim of this paper is to study the differentiability properties of the equilibrium prices and allocations for linear exchange economies with respect to the initial endowments and utility vectors. A linear economy means that the preferences of the agents can be represented by linear utility functions. These questions, which are related to sensitivity analysis, arise in a natural way in economic theory and they have been studied extensively since Debreu's work (Ref. 1); see for example Refs. 2–6 in the case of strictly quasiconcave utility functions. Hence, these results do not encompass the

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linear utilities functions. Moreover, in our case, we lose the uniqueness of the solution of the consumer maximization problem, which is fundamental together with the differentiability in the above quoted papers. Nevertheless, since we consider only linear utility functions, which have a simple representation, it is possible to develop a sensitivity analysis by introducing them as parameters of the economy.

Topological properties such as the upper and lower semicontinuity of the equilibrium price and allocation correspondences have been studied in Bonnisseau–Florig–Jofré (Ref. 7), where we also find a discussion of the interest and the applications of linear exchange economies. Furthermore, the sets of economies such that the equilibrium price (up to positive scale multiplication) or allocation is unique have been characterized. Here, we will concentrate on the set of strictly positive utility vectors and endowments in order to study the differentiability properties of both the equilibrium price and the equilibrium allocation mapping.

Although Cornet (Ref. 8) proved that an equilibrium is a solution of a convex program with the utility vectors and the initial endowments appearing as parameters, the sensitivity analysis tools coming from optimization cannot be applied. At present, the results assume some strong second-order information and sometimes the uniqueness of the optimal solution, which are not satisfied in our case. Besides this, our approach leads to locally explicit formulas for the equilibrium prices and allocations. This is also one of the reasons why we do not apply the approach of algebraic geometry using the algebraic structure of linear exchange economies (see Ref. 9).

By using the special properties of linear utility functions, we characterize an open dense subset of the space of utility vectors and initial endowments of full Lebesgue measure, called the space of regular economies. On the set of regular endowments, the equilibrium price vector is an infinitely differentiable function of the initial endowments if the utility functions are fixed. In the case where both the utility vectors and endowments vary, the result remains true if the equilibrium allocation is unique. The set of regular allocations with a unique equilibrium allocation is also open, dense, and of full Lebesgue measure. Then, one deduces that the equilibrium price is a locally Lipschitzian function on the whole space. These results are a consequence of a formula which gives explicitly the equilibrium price vector near a regular allocation. As a by product of this formula, we show that, if the endowment of one consumer in one good increases, then the relative equilibrium prices of the other goods increase or remain constant. In other words, this means that linear exchange economies satisfy a property of gross substitution. Cheng (Ref. 10) had already the intuition that linear exchange economies should satisfy this property.

Note that our purpose in this paper may be seen as the study of the equilibrium manifold, which is the graph of the mapping associating the equilibrium price vector to the parameters, i.e., the utility vectors and the initial endowments. Our results show that the equilibrium manifold is homeomorphic to the space of parameters and it is locally almost everywhere an infinitely differentiable submanifold. In the standard case (for strictly quasiconcave differentiable utility functions), the manifold is globally an infinitely differentiable submanifold of this space, but this is not true for linear economies.

In order to obtain the explicit formula mentioned above, we use extensively the fact that the equilibrium price vector is unique up to positive scale multiplication and its continuity with respect to the utility vectors and initial endowments. Furthermore, the solution of the consumer maximization problem, that is, the demand, has a particular structure, since it may be positive only for the commodities which give the maximum ratio between prices and marginal utilities. Consequently, with each equilibrium price vector, we can associate a correspondence defined on the set of consumers into the set of commodities, which plays an important role in the proof. Indeed, it allows us to find a finite partition of the set of utility vectors and initial endowments and to study the behavior of the equilibrium price vector on each component. The fact that the set of regular endowments is a full Lebesgue measure subset is a consequence of the semialgebraic structure of the equilibrium manifold.

In Section 2, we introduce the model and recall some properties of linear exchange economies. In Section 3, we establish explicit formulas for the equilibrium price mapping and the equilibrium allocation mapping in terms of the initial endowments and utility vectors. Section 4 is devoted to the proofs of the main results as a consequence of these formulas.

2. Model and Properties of Linear Exchange Economies

We consider a linear exchange economy with nonempty finite sets $L = \{1, \ldots, l\}$ of commodities and $I = \{1, \ldots, m\}$ of consumers. The consumption set of consumer i, $i \in I$, is R^l_+ and his utility function $u_i : R^l_+ \to R$ is defined by

$$u_i(x_i) = b_i \cdot x_i$$

for some given vector $b_i \in R^l_{++}$. His initial endowment is a vector ω_i in R^l_{++}. For each $(b, \omega) \in ((R^l_{++})^m)^2$, $\mathscr{L}(b, \omega)$ denotes the linear exchange economy associated to the parameters b and ω.

We now recall standard definitions and some known results. Their proofs can be found for example in either Bonnisseau–Florig–Jofré (Ref. 7), Cornet (Ref. 8), or Gale (Refs. 11–13).

Definition 2.1.

(i) If $p \in R^l_{++}$ is a price vector, the demand of the ith consumer, denoted $d(b_i, p, p \cdot \omega_i)$, is the set of solutions of the following maximization problem:

$$\max u_i(x_i) = b_i \cdot x_i,$$

$$p \cdot x_i \leq p \cdot \omega_i,$$

$$x_i \geq 0.$$

(ii) A Walras equilibrium of $\mathscr{S}(b, \omega)$ is an element $(x, p) \in (R^l_+)^m \times R^l_+$ such that:

(a) for every i, $x_i \in d(b_i, p, p \cdot \omega_i)$;
(b) $\sum_{i=1}^m x_i = \sum_{i=1}^m \omega_i$.

For every $(b, \omega) \in (R^l_{++})^m \times (R^l_{++})^m$, we denote by $X(b, \omega)$ the set of Walras equilibrium allocations in $(R^l_+)^m$ and by $P(b, \omega)$ the set of Walrasian equilibrium price vectors in R^l_{++}.

For consumer i, the marginal rate of substitution between the commodities h and k, denoted $r(b_i, h, k)$, is b_{ih}/b_{ik}. For each $p \in R^l_{++}$, let

$$\delta(b_i, p) = \{h \in \{1, \ldots, l\} | p_h \leq r(b_i, h, k) p_k, \forall k = 1, \ldots, l\}.$$

For $h \in \delta(b_i, p)$, the ratio between the marginal utility and the price is maximal. Thus, $\delta(b_i, p)$ is a set of commodities that consumer i wishes to consume if the price vector is p. Let ϵ^h be the vector of R^l whose coordinates are equal to 0, except the hth coordinate, which is equal to 1.

Proposition 2.1.

(i) For all $(b, \omega) \in ((R^l_{++})^m)^2$, for all $p \in R^l_{++}$, for each $i \in I$, $d(b_i, p, p \cdot \omega_i)$ is the convex hull of the points

$$((p \cdot \omega_i / p_h) \epsilon^h)_{h \in \delta(b_i, p)}.$$

(ii) The correspondence $X: ((R^l_{++})^m)^2 \to (R^l_+)^m$ is upper semicontinuous and has nonempty, convex, and compact values.

(iii) For all $(b, \omega) \in ((R^l_{++})^m)^2$, $P(b, \omega)$ is a half line of R^l_{++}, and if we choose a good h as numéraire, then the mapping $p: ((R^l_{++})^2 \to R^l_{++}$ defined by $p_h(b, \omega) = 1$ and $p(b, \omega) \in P(b, \omega)$ is continuous.

Note that the nonemptiness of the correspondence X is equivalent to the fact that the economy $\mathscr{L}(b, \omega)$ has an equilibrium. (iii) means that the equilibrium price vector is unique up to positive scale and is continuous when a normalization is chosen. Let v be the indirect utility function of the ith consumer; that is, the mapping from $R^l_{++} \times R^l_{++} \times R$ to R defined by

$$v(b_i, p, w_i) = w_i \max\{b_{ih}/p_h | h = 1, \ldots, l\}.$$

The uniqueness of the equilibrium price vector does not imply that $X(b, \omega)$ is reduced to a singleton. Nevertheless, for all $(b, \omega) \in ((R^l_{++})^m)^2$ and each i, the equilibrium utility level of consumer i is unique and is equal to $v(b_i, p(b, \omega), p(b, \omega) \cdot \omega_i)$.

We now introduce two subsets of $I \times L$, associated to the equilibrium of $\mathscr{L}(b, \omega)$, which are used extensively in the following. Let $p(b, \omega)$; then,

$$G(b, \omega) = \{(i, h) \in I \times L | h \in \delta(b_i, p(b, \omega))\},$$

$$G^+(b, \omega) = \{(i, h) \in I \times L | \exists x \in X(b, \omega), x_{ih} > 0\}.$$

One checks easily that $G(b, \omega)$ does not depend on the choice of $p(b, \omega)$ in $P(b, \omega)$. Since $X(b, \omega)$ is convex, there exists $x \in X(b, \omega)$ such that $x_{ih} > 0$ for all $(i, h) \in G^+(b, \omega)$. Proposition 2.1(i) implies that $G^+(b, \omega) \subset G(b, \omega)$. Furthermore, note that $G^+(b, \omega)$ [resp. $G(b, \omega)$] may be seen as a graph where the set of vertices is $I \times L$ and there exists an edge between the vertices i and h if and only if $(i, h) \in G^+(b, \omega)$ [resp. $(i, h) \in G(b, \omega)$].

The following proposition gives a necessary and sufficient condition for the uniqueness of the equilibrium allocation.

Proposition 2.2. Let (b, ω) in $((R^l_{++})^m)^2$. $X(b, \omega)$ is a singleton if and only if the graph $G^+(b, \omega)$ has no (nondegenerate) cycle; that is, there does not exist a finite family $((i_1, h_1), \ldots, (i_n, h_n))$ of two-by-two different elements of $I \times L$ such that (i_ν, h_ν) and $(i_\nu, h_{\nu+1})$ belong to $G^+(b, \omega)$, where $h_{n+1} = h_1$.

Let $\mathscr{B}$ be the set of elements $b \in (R^l_{++})^m$ such that, for all finite family of two-by-two different elements of $I \times L$, $((i_1, h_1), \ldots, (i_n, h_n))$,

$$\prod_{\nu=1}^{n} (b_{i_\nu h_\nu}/b_{i_\nu h_{\nu+1}}) \neq 1.$$

Then, Proposition 2.2 implies that, for all $(b, \omega) \in \mathscr{B} \times (R^l_{++})^m$, $X(b, \omega)$ is a singleton. Note that $\mathscr{B}$ is an open dense subset of $(R^l_{++})^m$.

The following lemma states the properties of the local behavior of the sets $G^+(b, \omega)$. The result depends on whether b is fixed or not. For

$y \in (R^l_{++})^m$ and $r > 0$, let

$$B(y, r) = \left\{ y' \in (R^l_{++})^m \,\middle|\, \sum_{i \in I} \|y_i - y'_i\| < r \right\}.$$

Lemma 2.1.

(i) For every (b, ω) in $((R^l_{++})^m)^2$, there exists $r > 0$ such that, for all $\omega' \in B(\omega, r)$, $G^+(b, \omega) \subset G^+(b, \omega')$.

(ii) For every (b, ω) in $((R^l_{++})^m)^2$, there exists $r > 0$ such that, for all $(b', \omega') \in (B(b, r) \times B(\omega, r))$, $G^+(b, \omega) \subset G^+(b', \omega')$ if and only if $X(b, \omega)$ is a singleton.

3. Explicit Formula for Equilibria

The computability of linear exchange economies has been studied before by Eaves (Ref. 14). He proposes a finite algorithm which computes, for any (b, ω), a Walras equilibrium $(x, p) \in X(b, \omega) \times P(b, \omega)$. Nevertheless, this method does not seen to allow for any conclusions on the behavior of the equilibrium price and the equilibrium allocation as a function of the fundamentals of the economy. In this section, we show that the equilibrium price vector can be obtained by an explicit formula which depends on the utility functions and the initial endowments. In the case where the equilibrium allocation is unique, we also show how to compute this allocation, when the equilibrium price vector is known. Actually, we exhibit a finite number of algebraic mappings and the equilibrium price vector is always given by one of these mappings. To choose the right one, it suffices to know the graph $G(b, \omega)$, or more precisely the connected components of this graph. For this purpose, we introduce some notations.

Let $\mathscr{C}$ be the finite set of the correspondences from I to L. With each economy (b, ω), we associate the correspondence C such that $G_C = G(b, \omega)$, where G_C is the graph of C. Conversely, for each $C \in \mathscr{C}$, let

$$\Omega^C = \{(b, \omega) \in ((R^l_{++})^m)^2 \mid G_C = G(b, \omega)\}.$$

We denote by $\bar{\mathscr{C}}$ the subset of the elements C of $\mathscr{C}$ such that Ω^C is nonempty. Note that $(\Omega^C)_{C \in \bar{\mathscr{C}}}$ is a finite partition of $((R^l_{++})^m)^2$.

Let us now consider a fixed element $C \in \bar{\mathscr{C}}$. We consider the graph with the set of vertices $I \times L$. There exists an edge between the vertices i and h if and only if $h \in C(i)$. We denote by $G^1_C, \ldots, G^n_C$ the connected components of this graph and by $\mathscr{I}^C_1, \ldots, \mathscr{I}^C_n$ [resp. $\mathscr{H}^C_1, \ldots, \mathscr{H}^C_n$] the projection of $G^1_C, \ldots, G^n_C$ on I [resp. L].

Since $C \in \bar{\mathscr{C}}$, there exists $(b, \omega) \in ((R^l_{++})^m)^2$ which is associated with C. Therefore, from the definitions of G_C and the definition of an equilibrium, $\mathscr{I}^C_1, \ldots, \mathscr{I}^C_n$ [resp. $\mathscr{H}^C_1, \ldots, \mathscr{H}^C_n$] is a partition of I [resp. L]. For each $z \in R^l$ and for each $\nu \in \{1, \ldots, n\}$, we denote by z^ν the restriction of z to the components in $\mathscr{H}^C_\nu$.

We obtain the formula in two steps. The following lemma shows that, if two economies are associated with the same graph, then the restrictions of the equilibrium price vectors to each subset of the partition of the set of commodities are proportional.

Lemma 3.1. For all $C \in \bar{\mathscr{C}}$, there exists an algebraic mapping π^C from $(R^l_{++})^m$ to R^l_{++} such that, for all $(b, \omega) \in \Omega^C$, for all $\nu = 1, \ldots, n$, $p^\nu(b, \omega)$ is proportional to $\pi^{C\nu}(b)$.

Proof. For each $\nu = 1, \ldots, n$, let $h^\nu \in \mathscr{H}^C_\nu$. From the definition of a connected component, for each $h \in \mathscr{H}^C_\nu$, there exists q consumers $i_1, \ldots, i_q$ and $q-1$ goods $h_1, \ldots, h_{q-1}$ such that, for each $k = 1, \ldots, q$,

$$h_{k-1} \in C(i_k) \quad \text{and} \quad h_k \in C(i_k),$$

where $h_0 = h^\nu$ and $h_q = h$.

We now define the mapping π^C as follows. For each $b \in (R^l_{++})^m$, for each $\nu = 1, \ldots, n$, for each $h \in \mathscr{H}^C_\nu$,

$$(\pi^C(b))_h = \begin{cases} 1, & \text{if } h = h^\nu, \\ \prod_{k=1}^q r_{i_k}(b_{i_k}, h_k, h_{k-1}), \text{ where } h_0 = h^\nu \text{ and } h_q = h, & \text{if } h \neq h^\nu. \end{cases}$$

Let $(b, \omega) \in \Omega^C$. Recalling the fact that $G_C = G(b, \omega)$ and recalling the definition of $\delta(b_i, p)$, for each $k = 1, \ldots, q$, we have

$$b_{i_k h_{k-1}} / p_{h_{k-1}}(b, \omega) = b_{i_k h_k} / p_{h_k}(b, \omega).$$

Thus, for all $\nu = 1, \ldots, n$,

$$p^\nu(b, \omega) = p_{h^\nu}(b, \omega) \pi^{C\nu}(b),$$

which ends the proof of the lemma. □

Now, we can state the main result of this section.

Proposition 3.1. For all $C\in\bar{\mathscr{C}}$, for all $(b,\omega)\in((R^l_{++})^m)^2$, let $T^C(b,\omega)$ be the $n\times n$ matrix defined by

$$t^C_{\nu\mu}(b,\omega)=\begin{cases}\pi^{C\nu}(b)\cdot\left(\sum_{i\notin\mathscr{I}^C_\nu}\omega^\nu_i\right), & \text{if } \nu=\mu,\\ -\pi^{C\mu}(b)\cdot\left(\sum_{i\in\mathscr{I}^C_\nu}\omega^\mu_i\right), & \text{if } \nu\neq\mu.\end{cases}$$

Then, for all $C\in\bar{\mathscr{C}}$, for all $(b,\omega)\in\Omega^C$, p is an equilibrium price vector of $\mathscr{S}(b,\omega)$ if and only if there exists $\lambda\in R^n_{++}$ in the kernel of $T^C(b,\omega)$ such that, for all $\nu\in\{1,\ldots,n\}$, $p^\nu=\lambda_\nu\pi^{C\nu}(b)$.

This result gives an explicit formula to compute the equilibrium price vector. Actually, when we have no information about the equilibrium price vector, this result gives a finite number of possibilities for the equilibrium price vector. This number may be large, since it is of the same level as the number of correspondences between I to L. Nevertheless, if we know the correspondence C which is associated with the economy $\mathscr{S}(b,\omega)$, that is, if we know the commodities desired by the consumers at equilibrium, then one has a unique formula.

Proof of Proposition 3.1. Let $(b,\omega)\in\Omega^C$. From Lemma 3.1, we deduce that there exists $\lambda(b,\omega)\in R^n_{++}$ such that

$$p^\nu(b,\omega)=\lambda_\nu(b,\omega)\pi^{C\nu}(b),$$

for all ν. We now show that $\lambda(b,\omega)$ belongs to the kernel of $T^C(b,\omega)$. For this, we use the Walras law (that is, the value of the equilibrium allocation of each consumer is equal to the value of his initial endowment) and the market clearing equation (that is, the sum of the equilibrium allocations is equal to the sum of the initial endowments). Furthermore, we use the fact that the equilibrium allocation of a consumer in the component $\mathscr{I}^C_\nu$ is positive only for the commodities in $\mathscr{H}^C_\nu$.

Let $(x,p(b,\omega))$ be an equilibrium of $\mathscr{S}(b,\omega)$. For all $i\in I$,

$$p(b,\omega)\cdot x_i=p(b,\omega)\cdot\omega_i.$$

Furthermore, recalling the fact that $G_C=G(b,\omega)$, one has, for all $\nu=1,\ldots,n$ and all $h\in\mathscr{H}^C_\nu$,

$$x_{ih}=0,\qquad \text{if } i\notin\mathscr{I}^C_\nu.$$

Consequently,

$$\sum_{i\in\mathscr{I}_\nu^C} x_{ih} = \sum_{i=1}^{m} \omega_{ih}.$$

For all $h \notin \mathscr{H}_\nu^C$,

$$\sum_{i\in\mathscr{I}_\nu^C} x_{ih} = 0.$$

Then, one deduces that

$$p(b,\omega)\cdot \sum_{i\in\mathscr{I}_\nu^C} x_i = p(b,\omega)\cdot \sum_{i\in\mathscr{I}_\nu^C} \omega_i = \sum_{\mu=1}^{n} \lambda_\mu(b,\omega)\pi^{C\mu}(b)\cdot\left(\sum_{i\in\mathscr{I}_\nu^C}\omega_i^\mu\right).$$

On the other hand,

$$\begin{aligned} p(b,\omega)\cdot \sum_{i\in\mathscr{I}_\nu^C} x_i &= p^\nu(b,\omega)\cdot \sum_{i\in\mathscr{I}_\nu^C} x_i^\nu \\ &= p^\nu(b,\omega)\cdot \sum_{i=1}^{m} \omega_i^\nu \\ &= \lambda_\nu(b,\omega)\pi^{C\nu}(b)\cdot \sum_{i=1}^{m}\omega_i^\nu. \end{aligned}$$

From the above equalities, one deduces that, for all $\nu = 1, \ldots, n$,

$$\lambda_\nu(b,\omega)\pi^{C\nu}(b)\cdot \sum_{i=1}^{m}\omega_i^\nu = \sum_{\mu=1}^{n}\lambda_\mu(b,\omega)\pi^{C\mu}(b)\cdot\left(\sum_{i\in\mathscr{I}_\nu^C}\omega_i^\mu\right),$$

or equivalently,

$$-\sum_{\mu\neq\nu}\lambda_\mu(b,\omega)\pi^{C\mu}(b)\cdot\left(\sum_{i\in\mathscr{I}_\nu^C}\omega_i^\mu\right)+\lambda_\nu(b,\omega)\pi^{C\nu}(b)\cdot\left(\sum_{i=1}^{m}\omega_i^\nu-\sum_{i\in\mathscr{I}_\nu^C}\omega_i^\nu\right)=0.$$

Since

$$\sum_{i=1}^{m}\omega_i^\nu - \sum_{i\in\mathscr{I}_\nu^C}\omega_i^\nu = \sum_{i\notin\mathscr{I}_\nu^C}\omega_i^\nu,$$

one has

$$T^C(b,\omega)\lambda(b,\omega) = 0.$$

Conversely, we remark that the rank of $T^C(b,\omega)$ is $n-1$, since the sum of the columns is zero and since the matrix $T^C_{\bar\nu}(b,\omega)$, which is the submatrix of $T^C(b,\omega)$ obtained by suppressing the $\bar\nu$th column and the $\bar\nu$th row is regular; see the proof of Corollary 3.1. Therefore, the kernel of $T^C(b, \omega)$ is a one-dimensional subspace of R^n. If λ is a positive element of the kernel of $T^C(b, \omega)$, then λ is positively proportional to $\lambda(b, \omega)$. Hence, the vector p defined by $p^\nu = \lambda^\nu \pi^{C\nu}(b)$ is positively proportional to $p(b, \omega)$. Consequently, p is an equilibrium price vector of $\mathscr{S}(b, \omega)$. $\square$

In the following corollary, we give a formula when a good is chosen as numéraire.

Corollary 3.1. Let $h \in \{1, \ldots, l\}$ be the commodity chosen as numéraire. For all $C \in \bar{\mathscr{C}}$, let $\bar{\nu} \in \{1, \ldots, n\}$ such that $h \in \mathscr{H}^C_{\bar{\nu}}$. Then, for all $(b, \omega) \in \Omega^C$, the equilibrium price vector $p(b, \omega)$ of $\mathscr{S}(b, \omega)$, which satisfies $p_h(b, \omega) = 1$, is given by the following formula:

$$p^{\bar{\nu}}(b, \omega) = [1/\pi^C_h(b)]\pi^{C\bar{\nu}}(b)$$

and, for all $\nu \neq \bar{\nu}$,

$$p^\nu(\omega) = \lambda_\nu(b, \omega)\pi^{C\nu}(b),$$

where $(\lambda_\nu(b, \omega))_{\nu \neq \bar{\nu}}$ is defined by

$$(\lambda_\nu(b, \omega))_{\nu \neq \bar{\nu}} = -[1/\pi^C_h(b)](T^C_{\bar{\nu}}(b, \omega))^{-1}(t^C_{\nu\bar{\nu}}(b, \omega))_{\nu \neq \bar{\nu}},$$

where the matrix $T^C_\nu(b, \omega)$ is the $(n-1) \times (n-1)$ submatrix of $T^C(b, \omega)$ obtained by suppressing the $\bar{\nu}$th column and the $\bar{\nu}$th row. Moreover, all the elements of the matrix $(T^C_\nu(b, \omega))^{-1}$ are nonnegative.

When a commodity is chosen as numéraire, the mapping which gives the unique equilibrium price vector is algebraic, since it is an algebraic combination of π^C, which is algebraic, and λ_ν, which is also algebraic since the matrix T^C [hence, $(T^C_\nu)^{-1}$] is algebraic. This can be deduced from general results of algebraic geometry. However, the explicit formula is not a by-product of this general approach.

Proof of Corollary 3.1. Let $T^C_{\bar{\nu}}(b, \omega)$ be the $(n-1) \times (n-1)$ submatrix of $T^C(b, \omega)$ obtained by suppressing the $\bar{\nu}$th column and the $\bar{\nu}$th row. First, we remark that the matrix $T^C_{\bar{\nu}}(b, \omega)$ is regular. Indeed, the elements on the diagonal are positive, the others elements are negative, and for all $\nu \neq \bar{\nu}$,

$$\sum_{\mu \neq \bar{\nu}} t^C_{\mu\nu}(b, \omega) = \pi^{C\nu}(b) \cdot \left(\sum_{i \notin \mathscr{I}^C_\nu} \omega^\nu_i - \sum_{\substack{\mu \neq \bar{\nu} \\ \mu \neq \nu}} \sum_{i \in \mathscr{I}^C_\mu} \omega^\nu_i \right)$$
$$= \pi^{C\nu}(b) \cdot \sum_{i \in \mathscr{I}^C_{\bar{\nu}}} \omega^\nu_i > 0.$$

Hence, the matrix $T_{\bar{v}}^C(b, \omega)$ is strictly diagonal dominant. Therefore, the system of equations given in the above statement has a unique solution. From Proposition 3.1, we know that the equilibrium price vector $p(b, \omega)$ satisfying $p_h(b, \omega) = 1$ is a solution. Consequently, $p(b, \omega)$ is the unique solution of the system.

Finally, a standard result on diagonal dominant matrices allows us to show that the elements of $(T_{\bar{v}}^C(b, \omega))^{-1}$ are nonnegative; see for example Ref. 15. □

In the following corollary, we extend the above result to the closure of Ω^C.

Corollary 3.2. Let $h \in \{1, \ldots, l\}$ be the commodity chosen as numéraire. For all $C \in \bar{\mathscr{C}}$, let $\bar{v} \in \{1, \ldots, n\}$ be such that $h \in \mathscr{H}_{\bar{v}}^C$. Then, for all $(b, \omega) \in \bar{\Omega}^C$, the closure of Ω^C in $((R_{++}^l)^m)^2$, the equilibrium price vector $p(b, \omega)$ of $\mathscr{L}(b, \omega)$ which satisfies $p_h(b, \omega) = 1$, is given by the following formula: $p^{\bar{v}}(b, \omega) = [1/\pi_h^C(b)]\pi^{C\bar{v}}(b)$ and, for all $v \neq \bar{v}$, $p^v(b, \omega) = \lambda_v(b, \omega)\pi^{Cv}(b)$, where $(\lambda_v(b, \omega))_{v \neq v}$ is defined by

$$(\lambda_v(b, \omega))_{v \neq \bar{v}} = -[1/\pi_h^C(b)](T_{\bar{v}}^C(b, \omega))^{-1}(t_{v\bar{v}}^C(b, \omega))_{v \neq \bar{v}}.$$

Proof. Let $(b, \omega) \in \bar{\Omega}^C$. There exists a sequence (b^q, ω^q) of Ω^C which converges to (b, ω). Since the equilibrium price vector is continuous, $(p(b^q, \omega^q))$ converges to $p(b, \omega)$. Furthermore, the matrix $(T_{\bar{v}}^C(\cdot))^{-1}$ is continuous with respect to (b, ω) and $p(b^q, \omega^q)$ is the unique solution of the linear system given in Corollary 3.1. Therefore, by a continuity argument, $p(b, \omega)$ is also the unique solution of the system of equations given in Corollary 3.2. □

From the above result, we can deduce the following global behavior of the equilibrium price vector.

Corollary 3.3. The mapping $p(\cdot, \cdot)$ is locally Lipschitzian on $((R_{++}^l)^m)^2$.

In the standard case, with strictly quasiconcave utility functions, the uniqueness of the equilibrium price vector implies a Lipschitzian behavior only if all the initial endowments are regular in the sense of differential geometry. Thus, the linear exchange economies have this special property, which allows us to use the tools of nonsmooth analysis in this framework like in Bonnisseau–Florig (Ref. 16).

Proof of Corollary 3.3. From Corollary 3.2, the mapping $p(\cdot,\cdot)$ is locally Lipschitzian on $\bar{\Omega}^C$ as the restriction of a nondegenerate algebraic mapping and $(\bar{\Omega}^C)_{C\in\bar{\mathscr{C}}}$ is a finite closed covering of $((R^l_{++})^m)^2$. Therefore, Corollary 3.3 is a direct consequence of the following result. Its proof is left to the reader. □

Lemma 3.2. Let U be an open subset of a finite-dimensional Euclidean space E. Let $F_1, \ldots, F_n$ be n closed (for the topology of U) subsets of U such that $U=\bigcup_{k=1}^n F_k$. For all $k=1,\ldots,n$, let f_k be a locally Lipschitzian mapping from F_k to a finite Euclidean space G. For all $x\in U$, let $K(x)=\{k\in\{1,\ldots,n\}\,|\,x\in F_k\}$. We assume that, for all $x\in U$, for all $k,k'\in K(x)$, $f_k(x)=f_{k'}(x)$. Let f be the mapping from U to G, defined by $f(x)=f_k(x)$ for some $k\in K(x)$. Then, f is locally Lipschitzian on U.

Now, we come to the computation of the equilibrium allocation when it is unique. If (b,ω) is such that $X(b,\omega)$ is a singleton, then from Proposition 2.2, the graph $G^+(b,\omega)$ has no cycle. Consequently, each connected component is a tree, and we shall use this property to find the equilibrium allocation. Note that, to compute the unique allocation $x\in X(b,\omega)$, it suffices to find x_{ih} for $(i,h)\in G^+(b,\omega)$, since the other components are equal to 0.

Let $\mathscr{S}$ be the set of element $(b,\omega)\in((R^l_{++})^m)^2$ such that $X(b,\omega)$ is a singleton. We consider the correspondence C such that its graph is $G^+(b,\omega)$.

Conversely, for each $C\in\mathscr{C}$, let

$$\Omega^C_+=\{(b,\omega)\in\mathscr{S}\,|\,G_C=G^+(b,\omega)\},$$

where G_C is the graph of C. We denote by $\bar{\mathscr{C}}_+$ the subset of the elements C of $\mathscr{C}$ such that Ω^C_+ is nonempty. Note that $(\Omega^C_+)_{C\in\bar{\mathscr{C}}_+}$ is a finite partition of $\mathscr{S}$.

Proposition 3.2. For all $C\in\bar{\mathscr{C}}_+$, there exists an algebraic mapping x^C from $R^l_{++}\times(R^l_{++})^m$ to $(R^l)^m$ such that, for each $(b,\omega)\in\Omega^C_+$, the unique element of $X(b,\omega)$ is $x^C(p(b,\omega),\omega)$.

Proof. Let $C\in\bar{\mathscr{C}}_+$, and let $G^1_C,\ldots,G^n_C$ be its connected components. From Proposition 2.2, each component has no cycle, hence it is a tree. We define the mapping x^C component-by-component and by induction on the number of elements in each component.

First, we let $x^C_{ih}(p,\omega)=0$ if (i,h) does not belong to the graph of C. If (i,h) is in the graph of C, then i and h belong to a component G^ν_C. If

$G_C^\nu = \{i, h\}$, this means that the ith consumer consumes only the hth commodity and he is the only one who consumes this commodity. In that case, we let

$$x_{ih}^C(p, \omega) = \sum_{i=1}^{m} \omega_{ih}.$$

If G_C^ν has more than two elements, then we associate the real number

$$w_i(p, \omega) = p \cdot \omega_i$$

to the vertex i and

$$w_h(p, \omega) = \sum_{i=1}^{m} \omega_{ih}$$

to the vertex h. Since G_C^ν is a tree, it has a terminal node. If this node is an element i of I, there exists $h \in L$ such that the edge (h, i) relies i to the tree. This means that consumer i consumes only the hth commodity. In that case, we let

$$x_{ih}^C(p, \omega) = w_i(p, \omega)/p_h.$$

Then, we consider the subtree obtained from G_C^ν be deleting the vertex i and the edge (h, i), and we replace $w_h(p, \omega)$ by $w_h(p, \omega) - w_i(p, \omega)/p_h$.

If the terminal node is an element h of L, there exists $i \in I$ such that the edge (i, h) relies h to the tree. This means that the hth commodity is consumed only by the ith consumer. In that case, we let

$$x_{ih}^C(p, \omega) = w_h(p, \omega),$$

we consider the subtree obtained from G_C^ν by deleting the vertex h and the edge (i, h), and we replace $w_i(p, \omega)$ by $w_i(p, \omega) - p_h w_h(p, \omega)$. In the two cases, one obtains a subtree with one less vertex. Consequently, in a finite number of steps, one defines a mapping x^C for each element (i, h) in the graph of C, which is algebraic since the operations are algebraic at each step.

To prove that $x^C(p(b, \omega), \omega)$ is the unique element of $X(b, \omega)$ for each $(b, \omega) \in \Omega_+^C$, it suffices to follow the steps which yield x^C and to check that the market clearing condition and the Walras law imply that, for each (i, h) in the graph of C, the equilibrium allocation x_{ih} is given by $x_{ih}^C(p(b, \omega), \omega)$. □

4. Regular Economies and Differentiability of the Equilibrium Price Vector

In this section, we define a class of economies, called regular economies, which have the nice property that, if the equilibrium allocation is unique, then the equilibrium price vector is a nondegenerate algebraic mapping (hence, real analytic) in a neighborhood of it. We prove that the set of economies which satisfy these conditions is open, dense, and of full Lebesgue measure.

Definition 4.1. The economy $\mathscr{L}(b, \omega)$ is regular if $G^{+}(b, \omega) = G(b, \omega)$.

From Proposition 2.1(i), we know that

$$x_{ih} = 0, \qquad \text{if } (i, h) \notin G(b, \omega).$$

An economy is regular if, at least for one equilibrium allocation, each consumer has a positive amount of all the commodities which lead to the best ratio between the marginal utility and the price. In the example given in Section 4 of Bonnisseau–Florig–Jofré (Ref. 7), we remark that some allocations satisfy this condition, but not all allocations. By Eaves (Ref. 14) and Bonnisseau–Florig–Jofré (Ref. 7), it is possible to construct $G^{+}(b, \omega)$ and $G(b, \omega)$ for any economy by applying a finite algorithm. For any economy, hence it is possible to check whether it is regular or not.

Let us define the equilibrium manifold $\mathscr{M} \subset ((R^{l}_{++})^{m})^{2} \times R^{l}_{++}$ as the graph of the mapping $(b, \omega) \rightarrow p(b, \omega)$ and, for each $b \in (R^{l}_{++})^{m}$, let

$$\mathscr{M}_{b} = \{(\omega, p(b, \omega)) \in (R^{l}_{++})^{m} \times R^{l}_{++} \mid (b, \omega, p(b, \omega)) \in \mathscr{M}\}.$$

We can consider the projection from $\mathscr{M}$ to $((R^{l}_{++})^{m})^{2}$ which associates (b, ω) with $(b, \omega, p(b, \omega))$ and the restriction of this projection at $\mathscr{M}_{b}$ when the utility vector b is fixed. Theorem 4.1 below implies that $\mathscr{M}_{b}$ is a C^{∞} submanifold in a neighborhood of $(\omega, p(b, \omega))$ when the economy $\mathscr{L}(b, \omega)$ is regular. Moreover, for a regular economy $\mathscr{L}(b, \omega)$, the vector ω is a regular value of the restriction of the projection in the standard sense of differential geometry. This explains why we use the word "regular" to define these economies just as in the papers dealing with strictly quasiconcave utility functions. If $\mathscr{L}(b, \omega)$ is regular and has a unique equilibrium allocation, then $\mathscr{M}$ is a C^{∞} submanifold in a neighborhood of $(b, \omega, p(b, \omega))$.

In the remainder of this section, we assume implicitly that one good is chosen as numéraire. Then, $p(b, \omega)$ denotes the unique equilibrium price vector of the economy $\mathscr{L}(b, \omega)$ which satisfies the normalization.

We now establish the first main result. We denote by Ω the set of elements (b, ω) such that $\mathscr{L}(b, \omega)$ is regular, by $\mathscr{S}$ the set of elements (b, ω) such that $X(b, \omega)$ is a singleton and, for every $b \in (R^{l}_{++})^{m}$, by Ω_{b} the set of elements $\omega \in (R^{l}_{++})^{m}$ such that $(b, \omega) \in \Omega$. In the following, we distinguish the cases where the utility vectors b_{i} are either fixed or not fixed.

Theorem 4.1.

(i) For all $(\bar{b}, \bar{\omega}) \in \Omega \cap \mathscr{S}$, there exists $r > 0$ such that $B(\bar{b}, r) \times B(\bar{\omega}, r) \subset \Omega \cap \mathscr{S}$, and, for all $(b, \omega) \in B(\bar{b}, r) \times B(\bar{\omega}, r)$,

$$G(b, \omega) = G(\bar{b}, \bar{\omega}).$$

The set $\Omega \cap \mathscr{S}$ is a semialgebraic open dense subset of $((R^l_{++})^m)^2$ and, consequently, $((R^l_{++})^m)^2 \backslash (\Omega \cap \mathscr{S})$ is a set of Lebesgue measure 0.

(ii) for all $\bar{b} \in (R^l_{++})^m$, for all $\omega \in \Omega_{\bar{b}}$, there exists $r > 0$ such that $B(\bar{\omega}, r) \subset \Omega_{\bar{b}}$ and, for all $\omega \in B(\bar{\omega}, r)$,

$$G(\bar{b}, \omega) = G(\bar{b}, \bar{\omega}).$$

The set $\Omega_{\bar{b}}$ is a semialgebraic open dense subset of $(R^l_{++})^m$ and, consequently, $(R^l_{++})^m \backslash \Omega_{\bar{b}}$ is a set of Lebesgue measure 0.

Now, we can state the results which give the local properties of the equilibrium price vector. In the following statement, we consider the correspondence X as a mapping, since $X(b, \omega)$ is a singleton on $\Omega \cap \mathscr{S}$.

Theorem 4.2.

(i) For each $(\bar{b}, \bar{\omega}) \in \Omega \cap \mathscr{S}$, let $r > 0$ as given by Theorem 4.1. Then, on $B(\bar{b}, r) \times B(\bar{\omega}, r)$, the mapping

$$(b, \omega) \rightarrow (p(b, \omega), X(p, \omega))$$

is the restriction of a nondegenerate algebraic mapping from $((R^l_{++})^m)^2$ to $R^l_{++} \times (R^l_{+})^m$. Consequently, the mapping $(p(\cdot,\cdot), X(\cdot,\cdot))$ is semialgebraic and real analytic on $\Omega \cap \mathscr{S}$.

(ii) For each $\bar{b} \in (R^l_{++})^m$, for all $\bar{\omega} \in \Omega_{\bar{b}}$, let $r > 0$ as given by Theorem 4.1. Then, on $B(\bar{\omega}, r)$, the mapping

$$\omega \rightarrow p(\bar{b}, \omega)$$

is the restriction of a nondegenerate algebraic mapping from $(R^l_{++})^m$ to R^l_{++}. Consequently, the mapping $p(\bar{b}, \cdot)$ is semialgebraic and real analytic on $\Omega_{\bar{b}}$.

The proof of this result is a direct consequence of Theorem 4.1, together with the explicit formula given in Proposition 3.1, Corollary 3.2, and Proposition 3.2. Indeed, the formulas given in Corollary 3.2 and

Proposition 3.2 remain true on $B(\bar{b}, r) \times B(\bar{\omega}, r)$ [resp. $B(\bar{\omega}, r)$], since

$$G(b, \omega) = G^{+}(b, \omega) = G(\bar{b}, \bar{\omega}),$$

$$[\text{resp. } G(\bar{b}, \omega) = G^{+}(\bar{b}, \omega) = G(\bar{b}, \bar{\omega})],$$

for each $(b, \omega) \in B(\bar{b}, r) \times B(\bar{\omega}, r)$ [resp. $\omega \in B(\bar{\omega}, r)$], so that $p(b, \omega)$ and $X(b, \omega)$ [resp. $p(\bar{b}, \omega)$] are the restrictions to $B(\bar{b}, r) \times B(\bar{\omega}, r)$ [resp. $B(\bar{\omega}, r)$] of algebraic mappings. This clearly implies that these mappings are real analytic on $\Omega \cap \mathscr{S}$ [resp. $\Omega_{\bar{b}}$].

These results may be compared to the ones given in Refs. 1–6 in which the main tools are coming from differential topology. They use the fact that the excess demand function is C^{∞} with respect to the prices. We cannot apply such results to a linear exchange economy, since the demand is not unique and is not differentiable with respect to the prices.

In the case of general smooth preferences, the implicit function theorem gives the existence of a function which corresponds to the equilibrium prices, but does not give an explicit formula. However, by using the particular structure of linear utility functions, we are able to obtain an explicit formula for the equilibrium prices. Furthermore, since the space of utility functions is parametrizable easily in the linear case, the above results take also into account the dependence of the equilibrium price vector with respect to the utility functions. In the standard case, one studies only the dependence with respect to the initial endowments.

In our framework, the fact that the equilibrium allocations have a nice behavior on $\Omega \cap \mathscr{S}$ is not a direct consequence of the behavior of the equilibrium price vector as in the standard case, since the demand correspondence is not a priori differentiable.

One deduces easily the structure of the equilibrium manifold from the above results. The continuity and the uniqueness of the mapping $p(\cdot, \cdot)$ imply that $\mathscr{M}$ [resp. $\mathscr{M}_b$] is homeomorphic to $((R^l_{++})^m)^2$ [resp. $(R^l_{++})^m$]. Furthermore, the image of the set $\Omega \cap \mathscr{S}$ [resp. Ω_b] by the mapping $(b, \omega) \rightarrow (b, \omega, p(b, \omega))$ [resp. $\omega \rightarrow (b, \omega, p(b, \omega))$] is an open dense subset of $\mathscr{M}$ [resp. $\mathscr{M}_b$], and $\mathscr{M}$ [resp. $\mathscr{M}_b$] is a $\mathscr{C}^{\infty}$ submanifold of $((R^l_{++})^m)^2 \times R^l_{++}$ [resp. $(R^l_{++})^m \times R^l_{++}$] around every point in this set.

Proof of Theorem 4.1. In the following, we give the arguments for part (i) and we just suggest the differences for part (ii), since they are almost the same. Let $(\bar{b}, \bar{\omega}) \in \Omega \cap \mathscr{S}$ [resp. $\bar{\omega} \in \Omega_{\bar{b}}$]. By the regularity of the initial endowment, one has

$$G^{+}(\bar{b}, \bar{\omega}) = G(\bar{b}, \bar{\omega}).$$

The continuity of the mapping $p(\cdot,\cdot)$ and the definition of $G(\cdot,\cdot)$ imply that there exists $\rho>0$ such that $G(b,\omega)\subset G(\bar{b},\bar{\omega})$ for all $(b,\omega)\in B(\bar{b},\rho)\times B(\bar{\omega},\rho)$ [resp. $\omega\in B(\bar{\omega},\rho)$]. From Lemma 2.1, one deduces that there exists $r\in]0,\rho[$ such that, for all $(b,\omega)\in B(\bar{b},r)\times B(\bar{\omega},r)$ [resp. $\omega\in B(\bar{\omega},r)$], $G^+(\bar{b},\bar{\omega})\subset G^+(b,\omega)$ [resp. $G^+(\bar{b},\bar{\omega})\subset G^+(\bar{b},\omega)$]. Consequently, for all $(b,\omega)\in B(\bar{b},r)\times B(\bar{\omega},r)$ [resp. $\omega\in B(\bar{\omega},r)$], one has

$$G^+(b,\omega)\subset G(b,\omega)\subset G(\bar{b},\bar{\omega})=G^+(\bar{b},\bar{\omega})\subset G^+(b,\omega)$$

$$[\text{resp. } G^+(\bar{b},\omega)\subset G(\bar{b},\omega)\subset G(\bar{b},\bar{\omega})=G^+(\bar{b},\bar{\omega})\subset G^+(\bar{b},\omega)].$$

Hence,

$$G^+(b,\omega)=G(b,\omega)=G(\bar{b},\bar{\omega})$$

$$[\text{resp. } G^+(\bar{b},\omega)=G(\bar{b},\omega)=G(\bar{b},\bar{\omega})].$$

This implies that $(b,\omega)\in\Omega$ [resp. $(\bar{b},\omega)\in\Omega_{\bar{b}}$]. Finally, for all $(b,\omega)\in B(\bar{b},r)\times B(\bar{\omega},r)$, $G^+(b,\omega)$ is locally constant; hence, from Proposition 2.2, $X(b,\omega)$ is a singleton, or equivalently, $(b,\omega)\in\mathscr{S}$. This ends part (i) of the proof.

Now, we come to the proof of the second part. The fact that Ω [resp. $\Omega_{\bar{b}}$] is an open subset of $((R^l_{++})^m)^2$ [resp. $(R^l_{++})^m$] is a direct consequence of the first part.

Now we show that $\Omega\cap\mathscr{S}$ is dense in $((R^l_{++})^m)^2$. For this, we use the fact that the set $\mathscr{B}$, defined after Proposition 2.2, is dense in $(R^l_{++})^m$. Let $(b,\omega)\in((R^l_{++})^m)^2$ and $r>0$. Then, there exists $b'\in B(b,r)\cap\mathscr{B}$. For $t>0$ small enough, ω^t defined by

$$\omega^t_i=\omega_i+t\sum_{h\,|\,(i,h)\in G(b',\omega)}\epsilon^h,\qquad \text{for all } i=1,\ldots,m,$$

belongs to $B(\omega,r)$. It suffices to show that $(b',\omega^t)\in\Omega\cap\mathscr{S}$. $X(b',\omega^t)$ is a singleton, since $b'\in\mathscr{B}$. Let $(x,p(b',\omega))$ be an equilibrium of $\mathscr{E}(b',\omega)$. Let $x^t\in(R^l_+)^m$ be defined by

$$x^t_i=x_i+t\sum_{h\,|\,(i,h)\in G(b',\omega)}\epsilon^h,\qquad \text{for all } i=1,\ldots,m,$$

One checks that $(x^t,p(b',\omega))$ is an equilibrium of $\mathscr{E}(b',\omega^t)$, since the allocations of the consumers increase for only the commodities which are desired for the price $p(b',\omega)$. Hence, $(b',\omega^t)\in\Omega$, since $x^t_{ih}>0$, for each $(i,h)\in G(b',\omega)=G(b',\omega')$.

To prove that $\Omega_{\bar{b}}$ is dense, it suffices to use the second part of the above argument. Using the notation of the previous section, let $C\in\bar{\mathscr{C}}$. To show that $\Omega\cap\mathscr{S}$ is semialgebraic, we use the following lemma.

Lemma 4.1. For each $C\in\bar{\mathscr{C}}$, Ω^C is a semialgebraic subset of $((R^l)^m)^2$.

The proof of this lemma is given in the Appendix. Note that this lemma implies that, for each $C \in \bar{\mathscr{C}}$, for each $\bar{b} \in (R^l_{++})^m$, the set

$$\Omega^C_{\bar{b}} = \{\omega \in (R^l_{++})^m \,|\, (\bar{b}, \omega) \in \Omega^C\}$$

is a semialgebraic subset of $(R^l_{++})^m$.

We now claim that

$$\Omega \cap \mathscr{S} = \bigcup_{C \in \bar{\mathscr{C}}} \operatorname{int} \Omega^C \left[\text{resp. } \Omega_{\bar{b}} = \bigcup_{C \in \bar{\mathscr{C}}} \operatorname{int} \Omega^C_{\bar{b}}\right].$$

Let $(\bar{b}, \bar{\omega}) \in \Omega \cap \mathscr{S}$ [resp. $\bar{\omega} \in \Omega_{\bar{b}}$], and let C such that $(\bar{b}, \bar{\omega}) \in \Omega^C$ [resp. $\bar{\omega} \in \Omega^C_{\bar{b}}$]. From the first part of the proof, there exists $r > 0$ such that, for all $(b, \omega) \in B(\bar{b}, r) \times B(\bar{\omega}, r)$ [resp. $\omega \in B(\bar{\omega}, r)$],

$$G(b, \omega) = G(\bar{b}, \bar{\omega}) \text{ [resp. } G(\bar{b}, \omega) = G(\bar{b}, \bar{\omega})].$$

Consequently, from the definition of Ω^C, $B(\bar{b}, r) \times B(\bar{\omega}, r) \subset \Omega^C$ [resp. $B(\bar{\omega}, r) \subset \Omega^C_{\bar{b}}$]; hence,

$$(\bar{b}, \bar{\omega}) \in \operatorname{int} \Omega^C \text{ [resp. } \bar{\omega} \in \operatorname{int} \Omega^C_{\bar{b}}].$$

Conversely, let $C \in \bar{\mathscr{C}}$, and let $(\bar{b}, \bar{\omega}) \in \operatorname{int} \Omega^C$. From the density of $\mathscr{B}$, one deduces that there exists $b' \in \mathscr{B}$ such that $(b', \bar{\omega}) \in \operatorname{int} \Omega^C$.

For $t > 0$, let ω^t be defined by

$$\omega^t_i = \bar{\omega}_i - t \sum_{h \,|\, (i,h) \in G_C} \epsilon^h, \qquad \text{for all } i = 1, \ldots, m.$$

For t small enough, $(\bar{b}, \omega^t)$ and (b', ω^t) belong to Ω^C. Let $(x^t, p(\bar{b}, \omega^t))$ be an equilibrium of $\mathscr{L}(\bar{b}, \omega^t)$, and let $(x'^t, p(b', \omega^t))$ be an equilibrium of $\mathscr{L}(b', \omega^t)$. Since

$$G_C = G(\bar{b}, \omega^t) = G(b', \omega^t),$$

one checks easily that

$$\left(\left(x_i = x^t_i + t \sum_{h \,|\, (i,h) \in G_C} \epsilon^h\right)_{i = 1, \ldots, m}, p(\bar{b}, \omega^t)\right)$$

is an equilibrium of $\mathscr{L}(\bar{b}, \bar{\omega})$ and

$$\left(\left(x'_i = x'^t_i + t \sum_{h \,|\, (i,h) \in G_C} \epsilon^h\right)_{i = 1, \ldots, m}, p(b', \omega^t)\right)$$

is an equilibrium of $\mathscr{L}(b', \bar{\omega})$. Therefore, for all $(i, h) \in G(\bar{b}, \bar{\omega})$, $x_{ih} > 0$, which implies that $(\bar{b}, \bar{\omega}) \in \Omega$. By the same argument, $(b', \bar{\omega})$ belongs to Ω. Since

$b' \in \mathscr{B}$, $(b', \bar{\omega})$ belongs also to $\mathscr{S}$. Consequently, from Proposition 2.2, $G^{+}(b', \bar{\omega})$ has no cycle. But since $(b', \bar{\omega})$ and $(\bar{b}, \bar{\omega})$ belong to Ω, one has

$$G^{+}(b', \bar{\omega}) = G(b', \bar{\omega}) = G_C = G(\bar{b}, \bar{\omega}) = G^{+}(\bar{b}, \bar{\omega}).$$

Hence, $G^{+}(\bar{b}, \bar{\omega})$ has no cycle, which implies that $(\bar{b}, \bar{\omega}) \in \mathscr{S}$.

To prove that an element $\bar{\omega} \in \operatorname{int} \Omega^C_{\bar{b}}$ belongs to $\Omega_{\bar{b}}$, it suffices to do the above argument with ω^t.

Now, we remark that

$$((R^l_{++})^m)^2 = \bigcup_{C \in \mathscr{C}} \Omega^C \left[\text{resp. } (R^l_{++})^m = \bigcup_{C \in \mathscr{C}} \Omega^C_{\bar{b}}\right].$$

A straightforward consequence of the stratification theorem for a semialgebraic set (see e.g. Ref. 17) is that $\operatorname{int} \Omega^C$ [resp. $\operatorname{int} \Omega^C_{\bar{b}}$)] is semialgebraic and the Lebesgue measure of $\Omega^C \backslash \operatorname{int} \Omega^C$ [resp. $\Omega^C_{\bar{b}} \backslash \operatorname{int} \Omega^C_{\bar{b}}$] is equal to 0. Thus, from the previous lemmas, $\Omega \cap \mathscr{S}$ [resp. $\Omega_{\bar{b}}$] is semialgebraic and the set $((R^l_{++})^m)^2 \backslash (\Omega \cap \mathscr{S})$ [resp. $R^l_{++} \backslash \Omega_{\bar{b}}$] is of a measure 0, since it is the disjoint union of the sets $\Omega^C \backslash \operatorname{int} \Omega^C$ [resp. $\Omega^C_{\bar{b}} \backslash \operatorname{int} \Omega^C_{\bar{b}}$]. Hence, $\Omega \cap \mathscr{S}$ [resp. $\Omega_{\bar{b}}$] is of full Lebesgue measure in $((R^l_{++})^m)^2$ [resp. $(R^l_{++})^m$], which ends the proof of Theorem 4.1. □

Now, we consider the case where the total initial endowment is fixed. For each $z \in R^l_{++}$, let E^z be the set defined by

$$E^z = \left\{\omega \in (R^l_{++})^m \,\middle|\, \sum_{i=1}^{m} \omega_i = z\right\}.$$

Then, we obtain the following corollary which shows that the results of Theorems 4.1 and 4.2 remain true if one restricts the initial endowments to E^z.

Corollary 4.1.

(i) For each $z \in R^l_{++}$, the set $(\Omega \cap \mathscr{S}) \cap ((R^l_{++})^m \times E^z)$ is a semialgebraic open dense subset of $((R^l_{++})^m \times E^z)$.

(ii) For every $\bar{b} \in (R^l_{++})^m$, for each $z \in R^l_{++}$, the set $\Omega_{\bar{b}} \cap E^z$ is a semialgebraic open dense subset of E^z.

In economic applications, we are interested often in a redistribution of the initial endowments among the consumers for a fixed total endowment. By applying the above result, one shows that such small redistribution induces a smooth variation of the equilibrium prices for almost all initial distributions.

Proof of Corollary 4.1. The only point to be proved is the density. Let $\omega \in E^z$, and let $\bar{b} \in R^l_{++}$. From the density of $\mathscr{B}$, there exists $b \in \mathscr{B}$ in each neighborhood of $\bar{b}$. Now, let $(x_1, \ldots, x_m, p(b, \omega))$ be an equilibrium of $\mathscr{S}(b, \omega)$. For all $h \in L$, let $i^h \in I$ such that $x_{i^h h} > 0$. Let $t_0 > 0$ such that

$$t_0 < \min\{x_{i^h h}, \omega_{i^h h} \,|\, h = 1, \ldots, l\}.$$

For all $h \in L$, let

$$I_h = \{i \in I \,|\, (i, h) \in G(b, \omega)\},$$

and let m_h be the cardinal of I_h. We remark that $i^h \in I_h$ and $m_h \leq m$. For all $t \in [0, 1]$, we consider the initial endowment ω^t defined by: for all $i \in I$, for all $h \in L$,

$$\omega^t_{ih} = \begin{cases} \omega_{ih}, & \text{if } (i, h) \notin G(b, \omega), \\ \omega_{ih} + tt_0/m, & \text{if } (i, h) \in G(b, \omega) \text{ and } i \neq i^h, \\ \omega_{ih} - (m_h - 1)tt_0/m, & \text{if } i = i^h. \end{cases}$$

We remark that

$$z = \sum_{i=1}^{m} \omega_i = \sum_{i=1}^{m} \omega^t_i$$

and that ω^t_i belongs to R^l_{++}. Furthermore, let $x^t \in (R^l_+)^m$ be defined by: for all $i \in I$, for all $h \in L$,

$$x^t_{ih} = \begin{cases} x_{ih}, & \text{if } (i, h) \notin G(b, \omega), \\ x_{ih} + tt_0/m, & \text{if } (i, h) \in G(b, \omega) \text{ and } i \neq i^h, \\ x_{ih} - (m_h - 1)tt_0/m, & \text{if } i = i^h. \end{cases}$$

One checks that $(x^t_1, \ldots, x^t_m, p(b, \omega))$ is an equilibrium of $\mathscr{S}(b, \omega^t)$ and that $\mathscr{S}(b, \omega^t)$ is a regular economy with a unique equilibrium. Since ω^t tends to ω when t converges to 0, one concludes that $(\Omega \cap \mathscr{S}) \cap (R^l_{++} \times E^z)$ is dense in $(R^l_{++} \times E^z)$, proving part (i).

Note that it suffices to do the same argument on the initial endowment without changing the utility vectors to prove part (ii) of the corollary. □

If we combine the explicit formula given in the previous section and the results of this section, we are able to determine the sign of the derivative of the equilibrium price of a commodity when the initial endowment of one consumer in another commodity changes. This leads to the following monotonicity property of the equilibrium prices.

Proposition 4.1. Let h be the commodity chosen as numéraire. For all $\bar{b}\in(R^l_{++})^m$, for all $\bar{\omega}\in\Omega_{\bar{b}}$, for all $i\in I$, and for all $k\in L$, $k\neq h$,

$$\frac{\partial p_k}{\partial\omega_{ih}}(\bar{b},\bar{\omega})\geq 0.$$

The above result does not depend on the commodity which is chosen as numéraire. It means that, if the initial endowment of one consumer in one commodity increases at a regular point, then the relative prices of the other commodities do not decrease.

Proof of Proposition 4.1. Let $\bar{b}\in R^l_{++}$, $\bar{\omega}\in\Omega_{\bar{b}}$, $i\in I$, and let $k\in L$, $k\neq h$. For $t\in R$, let ω^t be defined by $\omega^t_{i'}=\bar{\omega}_{i'}$, if $i'\neq i$, and $\omega^t_i=\bar{\omega}_i+t\epsilon^h$. For t close enough to zero, ω^t belongs to $B(\bar{\omega},r)$ as given by Theorem 4.1(ii). Let C be such that $(\bar{b},\bar{\omega})\in\Omega^C$, and let $\bar{\nu}$ be such that $h\in\mathscr{H}^C_{\bar{\nu}}$. We remark that the matrix $T^C_{\bar{\nu}}(\bar{b},\omega^t)$, as given by Corollary 3.2, does not depend on t. Moreover if $i\in\mathscr{I}^C_{\bar{\nu}}$, then

$$t_{\nu\bar{\nu}}(\bar{b},\omega^t)=t_{\nu\bar{\nu}}(\bar{b},\bar{\omega}).$$

Otherwise,

$$t_{\nu\bar{\nu}}(\bar{b},\omega^t)=t_{\nu\bar{\nu}}(\bar{b},\bar{\omega})-t\pi^C_h(\bar{b}).$$

Furthermore, since h is chosen as numéraire, the prices of the commodities in $\mathscr{H}^C_{\bar{\nu}}$ do not change and the prices of the other goods are increasing, since the elements of the matrix $(T^C_{\bar{\nu}}(\bar{b},\omega))^{-1}$ are all nonnegative. In the particular case where $i\in\mathscr{I}^C_{\bar{\nu}}$, nothing changes and the prices remain constant. This implies that the partial derivative $\partial_{p_k}/\partial\omega_{ih}(\bar{b},\bar{\omega})$ is nonnegative. □

Using the definition of the generalized gradient (Ref. 18), one can obtain a result which holds on the whole space $((R^l_{++})^m)^2$. Let $\partial p_k(b,\omega)$ be the generalized gradient of p_k at (b,ω) in the sense of Clarke, and let $\partial_\omega p_k(b,\omega)$ be the generalized gradient of $p_k(b,\cdot)$ at ω.

Corollary 4.2. Let h be the commodity chosen as numéraire. Let $i\in I$ and $k\in L$, $k\neq h$. For all $(\bar{b},\bar{\omega})\in((R^l_{++})^m)^2$,

$$u_{ih}\geq 0,\qquad \text{for all } u\in\partial p_k(\bar{b},\bar{\omega}).$$

Therefore, the mapping p_k is nondecreasing with respect to ω_{ih}. Furthermore,

$$u_{ih}\geq 0,\qquad \text{for all } u\in\partial_\omega p_k(\bar{b},\bar{\omega}).$$

In the economic terminology, this result means that a linear exchange economy satisfies a property of gross substitution. Cheng (Ref. 10) had already the intuition that linear exchange economies should satisfy this property. However, his proof depends for example on the claim that, at equilibrium, there are always agents consuming only one commodity. Of course, this is not true. Trivial counterexamples can be generated by splitting every commodity into two identical commodities.

Proof of Corollary 4.2. The first part of the corollary is a consequence of Proposition 4.1, Theorem 2.5.1 (Ref. 18, p. 63), since $\Omega \cap \mathscr{S}$ [resp. $\Omega_{\bar{b}}$] is an open dense subset of $((R^l_{++})^m)^2$ [resp. $(R^l_{++})^m$] of full Lebesgue measure. The second part is a consequence of the mean-value theorem (Ref. 18, p. 41). □

5. Appendix: Proof of Lemma 4.1

Let $C \in \bar{\mathscr{C}}$. For each $i \in I$, we choose arbitrarily an element $h_i \in C(i)$. We denote by $p^C(\cdot,\cdot)$ the algebraic mapping from $((R^l)^m)^2$ to R^l given by the formula of Corollary 3.2.

Let $(b,\omega) \in ((R^l)^m)^2$. If (b,ω) belongs to Ω^C, then

$$p^C(b,\omega) = p(b,\omega).$$

Therefore, there exists $x \in (R^l_+)^m$ such that the following system of equalities and inequalities holds true:

$$\omega_{ih} > 0 \quad \text{and} \quad b_{ih} > 0, \qquad \text{for all } (i,h) \in I \times L, \tag{1}$$

$$b_{ih_i}/p^C_{h_i}(b,\omega) = b_{ih}/p^C_h(b,\omega), \qquad \text{for all } i \in I, \text{ for all } h \in C(i), \tag{2}$$

$$b_{ih_i}/p^C_{h_i}(b,\omega) > b_{ih}/p^C_h(b,\omega), \qquad \text{for all } i \in I, \text{ for all } h \notin C(i), \tag{3}$$

$$x_{ih} = 0, \qquad \text{for all } (i,h) \notin G_C, \tag{4}$$

$$x_i \geq 0 \text{ and } p^C(b,\omega) \cdot x_i = p^C(b,\omega) \cdot \omega_i, \qquad \text{for all } i \in I, \tag{5}$$

$$\sum_{i \in I} x_i = \sum_{i \in I} \omega_i. \tag{6}$$

Indeed, (1) means that b_i and ω_i belong to R^l_{++} for all i; (2)–(3) are equivalent with $G_C = G(b,\omega)$; (4)–(5) are equivalent to $x_i \in d(b_i, p^C(b,\omega), p^C(b,\omega) \cdot \omega_i)$; and (6) implies that x clears the markets in each commodity.

Conversely, if (b,ω,x) satisfies (1)–(6), one deduces easily that $(x, p^C(b,\omega))$ is an equilibrium of $\mathscr{S}(b,\omega)$. Hence,

$$p^C(b,\omega) = p(b,\omega),$$

and (2)–(3) imply that $G_C = G(b,\omega)$, which is equivalent with $(b,\omega) \in \Omega^C$.

From above, one deduces that Ω^C is a semialgebraic subset of $(R^l)^m \times (R^l)^m$, since it is the projection on a set defined by a finite set of inequalities and equalities, which involve algebraic functions of (b, ω, x). $\square$

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