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Equilibrium Correspondence of Linear Exchange Economies[1]

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Abstract. According to Mertens (Ref. 1), the set of equilibrium prices in a linear exchange economy is a convex polyhedral cone (after adding $\{0\}$). We give a constructive proof of this fact. Then, we establish a lower-semicontinuity property of the equilibrium price correspondence. The set of equilibrium allocations is a closed, convex polyhedron. We give a characterization of this set.

Key Words. Walras equilibrium, linear utility functions, sensitivity analysis, computation, equilibrium correspondence.

1. Introduction

Linear exchange economies have been studied extensively (Refs. 1–9). As pointed out in Ref. 6, this model has been applied to a variety of different economic problems.

In the present paper, we give results on the computability of the set of Walras equilibria and we deduce a lower-semicontinuity property of the equilibrium price correspondence. Eaves (Ref. 5) proposed a finite algorithm computing a Walras equilibrium (provided an equilibrium exists) for any linear exchange economy. Mertens (Ref. 1, Theorem II.5) proved that the set of equilibrium prices in linear exchange economies is a convex polyhedral cone (after adding $\{0\}$). Relying on Ref. 5, we give a constructive proof of this fact. Then, we show that the equilibrium price correspondence stays lower semicontinuous as long as we do not perturb a certain subset of the

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support of the initial endowments of the consumers. Later, we characterize the set of equilibrium allocations.

2. Model

We consider a linear exchange economy with finite sets $L \equiv \{1, \ldots, L\}$ of commodities and $I \equiv \{1, \ldots, I\}$ of consumers. Every consumer is characterized by his utility function u_i: $\mathbb{R}_+^L \rightarrow R$, which is defined by $u_i(x_i) = b_i \cdot x_i$ for a given vector $b_i \in \mathbb{R}_+^L$, and by his initial endowment $\omega_i \in \mathbb{R}_+^L$. For each $(b, \omega) \in (\mathbb{R}_+^L)^{2I}$, $\mathscr{L}(b, \omega)$ denotes the linear exchange economy associated with the parameters b and ω. Throughout the paper, we will make the following assumptions:

(A) $(\sum_{i \in I} b_i, \sum_{i \in I} \omega_i) \in \mathbb{R}_{++}^L \times \mathbb{R}_{++}^L$;
(B) for every i, $b_i \neq 0$, $\omega_i \neq 0$.

Condition (A) means that every good is desired at least by some consumer and owned at least by a consumer and Condition (B) means that every consumer desires at least one good and owns at least one good.

Definition 2.1.

(i) For $p \in \mathbb{R}^L$, the demand of consumer i, denoted $d(b_i, p, p \cdot \omega_i)$, is the set of solutions of the following maximization problem:

$$\begin{aligned} \max \quad & u_i(x_i) = b_i \cdot x_i, \\ \text{s.t.} \quad & p \cdot x_i \leq p \cdot \omega_i, \\ & x_i \in \mathbb{R}_+^L. \end{aligned}$$

(ii) A Walras equilibrium of $\mathscr{L}(b, \omega)$ is an element $(x, p) \in (\mathbb{R}_+^L)^I \times \mathbb{R}^L$ such that:

(a) for every i, $x_i \in d(b_i, p, p \cdot \omega_i)$;
(b) $\sum_{i \in I} x_i = \sum_{i \in I} \omega_i$.

(iii) A proper subset I' of I is called self-sufficient in $\mathscr{L}(b, \omega)$ if, for all $h \in L$, $\sum_{i \in I'} b_{ih} > 0$ implies $\sum_{i \in I \setminus I'} \omega_{ih} = 0$.

(iv) A proper subset I' of I is called super self-sufficient in $\mathscr{L}(b, \omega)$ if it is self-sufficient and there exists $h \in L$ such that $\sum_{i \in I'} \omega_{ih} > 0$, but $\sum_{i \in I'} b_{ih} = 0$.

A subset of the set of traders is self-sufficient, if they own the whole quantity of goods they are interested in and it is called super-self-sufficient, if they own as well a positive amount of some good which nobody in their group is interested in.

Let $\mathscr{W} \subset (\mathbb{R}^L_+)^I \times (\mathbb{R}^L_+)^I$ denote the set of pairs (b, ω) such that no super-self-sufficient subset exists in $\mathscr{L}(b, \omega)$ and such that conditions (A) and (B) are satisfied. Under Assumptions (A) and (B), the nonexistence of a super-self-sufficient set is a necessary and sufficient condition for the existence of a Walras equilibrium in linear exchange economies (Ref. 4). We denote by $\mathscr{U}$ the set of pairs $(b, \omega) \in \mathscr{W}$ such that $\mathscr{L}(b, \omega)$ has a unique equilibrium price vector $p(b, \omega)$ up to a positive scale multiplication.

For every $(b, \omega) \in (\mathbb{R}^L_+)^I \times (\mathbb{R}^L_+)^I$, we denote by $X(b, \omega) \subset (\mathbb{R}^L_+)^I$ the set of Walrasian equilibrium allocations and by $P(b, \omega) \subset \mathbb{R}^L$ the set of Walrasian equilibrium price vectors. By Assumption (B), $P(b, \omega) \subset \mathbb{R}^L_{++}$.

For the ith consumer, the marginal rate of substitution between the commodities h and k is b_{ih}/b_{ik}, where by convention $0/0 = 0$ and $b_{ih}/0 = +\infty$ if $b_{ih} > 0$. For each $p \in \mathbb{R}^L_{++}$,

$$\delta(b_i, p) = \{h \in L | p_h \leq p_k(b_{ih}/b_{ik}),\ \forall k \in L\}.$$

$\delta(b_i, p)$ is the set of commodities that the consumer wishes to consume if the price vector is p, since the ratio between the marginal utility and the price is maximal for these commodities. For $y \in \mathbb{R}^n$, we denote by

$$\mathrm{supp}(y) = \{h \in \{1, \ldots, n\} | y_h \neq 0\}$$

the support of y.

3. Computation of Equilibrium Prices

We recall first a necessary and sufficient condition for $P(b, \omega)$ being a half line. It is proven in Ref. 7.

Proposition 3.1. Let $(b, \omega) \in \mathscr{W}$ and let $p \in P(b, \omega)$. Then, $(b, \omega) \in \mathscr{U}$ if and only if the economy $\mathscr{L}(c(p), \omega)$, defined by

$$c_{ih}(p) = \begin{cases} b_{ih}, & \text{if } h \in \delta(b_i, p), \\ 0, & \text{otherwise}, \end{cases}$$

has no proper self-sufficient subset.

Partition of the Economy. We will construct a partition $I_1, \ldots, I_k$ of I and $L_1, \ldots, L_k$ such that $P(b, \omega)$ is generated by the equilibrium prices of the subeconomies restricted to $I_r \times L_r$, which are unique up to a positive scale multiplication.

By applying Ref. 5, it is possible to compute a Walras equilibrium $(x,p)\in X(b,\omega)\times P(b,\omega)$ in a finite number of steps. Now, $P(b,\omega)$ may be computed as follows.

Let $c=c(p)$ be as in Proposition 3.1. Let $I_1^1,\dots,I_{k_1}^1$ be minimal self-sufficient subsets of $\mathscr{L}(c,\omega)$. For all $r\geq 1$, let $I_1^r,\dots,I_{k_r}^r$ be minimal self-sufficient subsets of the economy $\mathscr{L}^r(c,\omega)$, which is obtained from $\mathscr{L}(c,\omega)$ by restricting the economy to

$$I\Bigg\backslash\left\{\bigcup_{s=1}^{r-1} I^s\right\}\times L\Bigg\backslash\left\{\bigcup_{s=1}^{r-1} L^s\right\},$$

where

$$I^s = I_1^s\cup\cdots\cup I_{k_s}^s \quad \text{and} \quad L^s = \bigcup_{i\in I^s}\operatorname{supp}\omega_i.$$

Let

$$\tilde{k} = \min\left\{ r\in\mathbb{N} \,\middle|\, L = \bigcup_{s=1}^{r} L^s\right\}.$$

Let $k=\sum_{r=1}^{\tilde{k}} k_r$ and let $I_1,\dots,I_k$ be a relabelling of the sets $I_1^1,\dots,I_{k_1}^1,\dots,$ $I_1^{\tilde{k}},\dots,I_{k_{\tilde{k}}}^{\tilde{k}}$ with $I_{\rho+\sum_{r'=1}^{r-1}k_{r'}}=I_r^\rho$. For all $r\in\{1,\dots,k\}$, let $L_r=\bigcup_{i\in I_r}\operatorname{supp}\omega_i$. Our relabelling is chosen such that, for all $r\in\{1,\dots,k\}$, for all $i\in\bigcup_{\rho\leq r}I_\rho$, and all $h\in\bigcup_{\rho>r}L_\rho$, $h\notin\delta(b_i,p)$.

For any vector $z\in\mathbb{R}^L$, let $z_{|L_r}$ be the restriction of z to L_r and let z_r be the canonical injection of $z_{|L_r}$ into $\mathbb{R}^L$. Moreover, let $\mathscr{L}_{|I_r\times L_r}(b,\omega)$ [resp. $\mathscr{L}_{|I_r}(b,\omega)$] be the economy $\mathscr{L}(b,\omega)$ obtained by restricting $I\times L$ to $I_r\times L_r$ [resp. I_r].

Lemma. 3.1. For all $r\in\{1,\dots,k\}$, for all $x\in X(b,\omega)$, $\operatorname{supp}x_i\subset L_r$ and $p_{|L_r}$ is the unique Walras equilibrium price of the economy $\mathscr{L}_{|I_r\times L_r}(b,\omega)$ up to a positive scale multiplication.

Proof. For all $r\in\{1,\dots,k\}$, for all $i\in I_r$, $\operatorname{supp}\omega_i\subset L_r$. By Proposition 3.1(i) in Ref. 7, for all $x\in X(b,\omega)$, (x,p) is a Walras equilibrium. Since for all $i\in I_1$, $\delta(b_i,p)\subset L_1$, we have $\operatorname{supp}x_i\subset L_1$. This implies that

$$\sum_{i\in I_1}x_i = \sum_{i\in I_1}\omega_i = \sum_{i\in I}\omega_{i1}.$$

Thus, for all $i\notin I_1$, $\operatorname{supp}x_i\cap L_1=\emptyset$. Since for all $r\in\{1,\dots,k\}$, for all $i\in I_r$,

$$\delta(b_i,p)\subset\bigcup_{\rho\leq r}L_\rho,$$

we have by induction that, for all $r \in \{1, \ldots, k\}$, for all $i \in I_r$,

$$\operatorname{supp} x_i \subset L_r.$$

Now, it is easy to check that $(x_{|I_r \times L_r}, p_{|L_r})$ is a Walras equilibrium of $\mathscr{L}_{|I_r \times L_r}(b, \omega)$. Since the only self-sufficient set in $\mathscr{L}_{|I_r \times L_r}(b, \omega)$ is I_r, $p_{|L_r}$ is the unique Walras equilibrium price up to a positive scale multiplication. $\square$

The next proposition and corollary follows closely Theorem II.5 of Ref. 1. Our proofs are straightforward adaptations of parts of Theorem II.5 and Corollary II.6 in Ref. 1. Moreover, the partition that we use will turn out to be identical to the partition in Theorem II.5 of Ref. 1. However, this will be established only once we have proved the following proposition.

For every $r \in \{1, \ldots, k\}$ and every $i \in I$, let

$$\alpha_r(i) = \max_{h \in L_r} b_{ih}/p_h.$$

For every pair $(r', r) \in \{1, \ldots, k\}^2$, let

$$\theta_{r'r} = \max_{i \in I_r} \alpha_{r'}(i)/\alpha_r(i).$$

This is well defined since, for every $r \in \{1, \ldots, k\}$ and every $i \in I_r$, $\alpha_r(i) > 0$. Let

$$\Lambda = \{\lambda \in \mathbb{R}^k_{++} \,|\, \lambda_{r'} \geq \theta_{r'r}\lambda_r, \forall (r', r) \in \{1, \ldots, k\}^2\}.$$

It is easy to see that $\Lambda \cup \{0\}$ is a convex polyhedral cone. Moreover, if for all $(r', r) \in \{1, \ldots, k\}^2$, $\theta_{r'r} > 0$, then $\Lambda \cup \{0\}$ is closed.

Proposition 3.2. Let $(b, \omega) \in \mathscr{W}$. Then,

$$P(b, \omega) = \left\{ \sum_{r=1}^{k} \lambda_r p_r \,\middle|\, \lambda \in \Lambda \right\}.$$

The proof of Proposition 3.2 is based on the following claims.

Claim 3.1. Let $q \in P(b, \omega)$. Then, there exists $\lambda \in \mathbb{R}^k_{++}$ such that $q = \sum_{r=1}^k \lambda_r p_r$.

Proof of Claim 3.1. Note first that, for all $q \in P(b, \omega)$ and for all $r \in \{1, \ldots, k\}$, $q_{|L_r}$ is a Walras equilibrium price of the economy $\mathscr{L}_{|I_r \times L_r}(b, \omega)$, which is obtained from $\mathscr{L}(b, \omega)$ by restricting $I \times L$ to $I_r \times L_r$.

The economy $\mathscr{L}_{|I_r \times L_r}(c, \omega)$, with c as in Proposition 3.1, has no proper self-sufficient subset. Thus, by Proposition 3.1, $p_{|L_r}$ is the only equilibrium price of the economy $\mathscr{L}_{|I_r \times L_r}(b, \omega)$ up to a positive scale multiplication.

Therefore, q_r is collinear to p_r. Then, every $q \in P(b, \omega)$ is of the form $q = \sum_{r=1}^{k} \lambda_r p_r$ for some $\lambda \in \mathbb{R}_{++}^k$. □

Claim 3.2. Let $q \in P(b, \omega)$. Then, there exists $\lambda \in \Lambda$ such that $q = \sum_{r=1}^{k} \lambda_r p_r$.

Proof of Claim 3.2. Let $q = \sum_{r=1}^{k} \lambda_r p_r \in P(b, \omega)$, with $\lambda \in \mathbb{R}_{++}^k$ and $\lambda \notin \Lambda$. Thus, there exists $(r', r) \in \{1, \ldots, k\}^2$ such that $\lambda_{r'} < \theta_{r'r} \lambda_r$. Let

$$j \in \arg\max_{i \in I_r} (\alpha_{r'}(i)/\alpha_r(i)),$$

$$h_r \in \arg\max_{h \in L_r} (b_{jh}/p_h).$$

By Lemma 3.2, we may have chosen $h_r \in L_r$ such that, for some $x \in X(b, \omega)$, $x_{jh_r} > 0$. Thus, since by Proposition 3.1(i) in Ref. 7, (x, q) is a Walras equilibrium $\mathscr{L}(b, \omega)$, we have $h_r \in \delta(b_j, q)$. Let

$$h_{r'} \in \arg\max_{h \in L_{r'}} (b_{jh}/p_h).$$

Since

$$b_{jh_{r'}}/\lambda_{r'} p_{h_{r'}} > b_{jh_{r'}}/\theta_{r'r} \lambda_r p_{h_{r'}}$$

and since

$$\theta_{r'r} = \alpha_{r'}(j)/\alpha_r(j) = (b_{jh_{r'}}/p_{h_{r'}})/(b_{jh_r}/p_{h_r}),$$

thus,

$$b_{jh_{r'}}/\lambda_{r'} p_{h_{r'}} > b_{jh_r}/\lambda_r p_{h_r}.$$

Then, $h_r \notin \delta(b_j, q)$ and thus $q \notin P(b, \omega)$, yielding a contradiction. □

Claim 3.3. Let $\lambda \in \Lambda$ and $q = \sum_{r=1}^{k} \lambda_r p_r$. Then, $q \in P(b, \omega)$.

Proof of Claim 3.3. Let $x \in X(b, \omega)$. By Lemma 3.2, for every $r \in \{1, \ldots, k\}$ and for every $i \in I_r$, $\operatorname{supp} x_i \subset L_r$ and $\operatorname{supp} \omega_i \subset L_r$. Thus, for every $i \in I$,

$$q \cdot x_i = q \cdot \omega_i.$$

It remains only to prove that, for all $r \in \{1, \ldots, k\}$ and for every $i \in I_r$, $\operatorname{supp} x_i \subset \delta(b_i, q)$. Let $r \in \{1, \ldots, k\}$, $i \in I_r$; and let $h \in \operatorname{supp} x_i \subset L_r$. We will prove that $h \in \delta(b_i, q)$. Let $h' \in L_r$. Then,

$$b_{ih}/p_h \geq b_{ih'}/p_{h'},$$

since (x, p) is an equilibrium and therefore,

$$b_{ih}/q_h \geq b_{ih'}/q_{h'}.$$

Let $h' \in L_{r'}$. Since

$$\theta_{r'r} \geq (b_{ih'}/p_{h'})/(b_{ih}/p_h)$$

and $0 < \theta_{r'r}\lambda_r \leq \lambda_{r'}$, we have

$$\theta_{r'r} b_{ih}/\theta_{r'r}\lambda_r p_h \geq b_{ih'}/\lambda_{r'} p_{h'},$$

and therefore,

$$b_{ih}/q_h \geq b_{ih'}/q_{h'}.$$

Hence,

$$h \in \delta(b_i, q). \qquad \square$$

Proof of Proposition 3.2. By Claim 3.2,

$$P(b, \omega) \subset \left\{ \sum_{r=1}^{k} \lambda_r p_r \,\middle|\, \lambda \in \Lambda \right\},$$

and by Claim 3.3,

$$\left\{ \sum_{r=1}^{k} \lambda_r p_r \,\middle|\, \lambda \in \Lambda \right\} \subset P(b, \omega). \qquad \square$$

Corollary 3.1. For all $(r, r') \in \{1, \ldots, k\}^2$ and for all $(h, h') \in L_r \times L_{r'}$, let $\beta_{hh'} = \theta_{rr'} p_h / p_{h'}$. Then,

$$P(b, \omega) = \{q \in \mathbb{R}^L_{++} \,|\, q_h \geq \beta_{hh'} q_{h'}, \forall h, h' \in L\}.$$

Proof. Let $q \in P(b, \omega)$. Then, by the previous proposition, there exists $\lambda \in \Lambda$ such that $q = \sum_{r=1}^{k} \lambda_r p_r$. Let $(r, r') \in \{1, \ldots, k\}^2$ and let $(h, h') \in L_r \times L_{r'}$. If $r = r'$, then since $\theta_{rr} = 1$,

$$\beta_{hh'} = p_h / p_{h'}$$

and thus,

$$q_h = \lambda_r p_h = (p_h / p_{h'}) \lambda_r p_{h'} = \beta_{hh'} q_{h'}.$$

Thus, it remains to check that the constraints hold also for $r \neq r'$. As $\lambda \in \Lambda$,

$$q_h = \lambda_r p_h \geq \theta_{rr'} \lambda_{r'} p_h = (\theta_{rr'} p_h / p_{h'}) \lambda_{r'} p_{h'} = \beta_{hh'} q_{h'}.$$

For the converse, let

$$\pi \in \{q \in \mathbb{R}^L_{++} | q_h \geq \beta_{hh'} q_{h'}, \forall h, h' \in L\},$$

with $\beta_{hh'} = \theta_{rr'}p_h/p_{h'}$ such that $h \in L_r$ and $h' \in L_{r'}$. Let $(h, h') \in L_r \times L_{r'}$. Then, if $r = r'$, since $\theta_{rr} = 1$,

$$\pi_h \geq (p_h/p_{h'})\pi_{h'}, \qquad \pi_{h'} \geq (p_{h'}/p_h)\pi_h.$$

Thus,

$$\pi_{h'}/p_{h'} = \pi_h/p_h.$$

Therefore, π is of the form $\sum_{r=1}^{k} \lambda_r p_r$ with $\lambda_r = \pi_h/p_h$ for any $h \in L_r$. Now, suppose that $r \neq r'$. Then,

$$\lambda_r p_h = \pi_h \geq (\theta_{rr'}p_h/p_{h'})\pi_{h'} = (\theta_{rr'}p_h/p_{h'})\lambda_{r'}p_{h'}.$$

Hence,

$$\lambda_r \geq \theta_{rr'}\lambda_{r'}$$

and therefore,

$$\pi = \sum_{r=1}^{k} \lambda_r p_r, \qquad \text{with } \lambda \in \Lambda. \qquad \square$$

Proposition 3.3. Let $(b, \omega) \in \mathscr{W}$ and let $L_1, \ldots, L_k$ be a partition of L constructed as above with respect to $p \in P(b, \omega)$. Then, the partition is the coarsest one of L such that, for all $q \in P(b, \omega)$, the function $\ell_q: L \to \mathbb{R}$, defined by $\ell_q(h) = q_h/p_h$ for all $h \in L$, is measurable.

Proof. Note first that, for all $r, r' \in \{1, \ldots, k\}$, we have $\theta_{r'r} \leq 1$. Thus, $(1, \ldots, 1) \in \Lambda$. Moreover, for all $i \in \bigcup_{\rho \leq r} I_\rho$ and all $h \notin \bigcup_{\rho \leq r} I_\rho$, $h \notin \delta(b_i, p)$; thus, if $r < r'$, then $\theta_{r'r} < 1$. For all $\epsilon > 0$ and $r \in \{1, \ldots, k\}$, let $\lambda^\epsilon(r) \in \mathbb{R}^k$ be defined by

$$\lambda^\epsilon_\rho(r) = 1 + \epsilon, \qquad \text{for all } \rho \leq r,$$

$$\lambda^\epsilon_\rho(r) = 1, \qquad \text{for all } \rho > r.$$

For small enough $\epsilon > 0$, $\lambda^\epsilon(r) \in \Lambda$.

If our partition were not the coarsest, then for some $r, r' \in \{1, \ldots, k\}$ with $r < r'$, all ℓ_q would be measurable with respect to the partition obtained from $L_1, \ldots, L_k$ replacing L_r and $L_{r'}$ by their union. Thus, for all small enough $\epsilon > 0$,

$$p^\epsilon = \sum_{\rho \leq r} \lambda^\epsilon_\rho(r) p_\rho + \sum_{\rho > r} p_\rho \in P(b, \omega).$$

However, ℓ_{p^ϵ} is not measurable with respect to this coarser partition. $\square$

Therefore, the partition generated above is the same as in Theorem II.5 of Ref. 1. This proves (by Ref. 1, page 21, lines 1–3) that our partition does not depend on the particular choice of $p \in P(b, \omega)$. Reference 1 proved, using the coarsest partition making all the above functions measurable, that the set $P(b, \omega)$ is a polyhedral cone generated by the canonical injections of the vectors $p_{|L_r}$ into $\mathbb{R}^L$. Not knowing $P(b, \omega)$, it would be hard of course to know the partition, which in turn we have to know to compute the set $P(b, \omega)$.

4. Lower Semicontinuity of Prices

The upper and lower semicontinuity of X and the upper semicontinuity of P has been studied in Ref. 7. We will study now, when P restricted to some subset of $\mathscr{W}$ is lower semicontinuous.

Let $(\bar{b}, \bar{\omega}) \in \mathscr{W}$ and let $(I_r)_{r=1}^k$ and $(L_r)_{r=1}^k$ be the corresponding partition of the economy. Let

$$\mathscr{M}(\bar{b}, \bar{w}) = \{(b, \omega) \in \mathscr{W} | \forall r \in \{1, \ldots, k\}, \forall i \in I_r, \forall h \notin L_r, \omega_{ih} = 0\}.$$

Of course, $(\bar{b}, \bar{\omega}) \in \mathscr{M}(\bar{b}, \bar{\omega})$. For $W \subset \mathscr{W}$, let $P: W \to \mathbb{R}_{++}^L$ be the restriction of the equilibrium price correspondence $P: \mathscr{W} \to \mathbb{R}_{++}^L$ to the set W.

Proposition 4.1. For each $(\bar{b}, \bar{\omega}) \in \mathscr{W}$, the correspondence

$$P: \mathscr{M}(\bar{b}, \bar{\omega}) \to \mathbb{R}_{++}^L$$

is lower semicontinuous at $(\bar{b}, \bar{\omega})$.

Proof. Suppose that the proposition is not true. Then, there exists an open set $V \subset \mathbb{R}^L$ such that $P(\bar{b}, \bar{\omega}) \cap V \neq \emptyset$ and, for a sequence $(b^n, \omega^n) \subset \mathscr{M}(\bar{b}, \bar{\omega})$, converging to $(\bar{b}, \bar{\omega})$ as n goes to $+\infty$, we have

$$P(b^n, \omega^n) \cap V = \emptyset, \qquad \text{for every } n.$$

We may assume p to be in the relative interior of $P(\bar{b}, \bar{\omega}) \cap V$. By Theorem II.5.2 of Ref. 1, Λ is of full dimension.[3] Thus, for all $r \in \{1, \ldots, k\}$, for all small enough $\epsilon > 0$,

$$p + \epsilon p_r \in P(b, \omega).$$

[3]This can be derived also from the fact that, for all small enough $\epsilon > 0$, for all r, $\lambda^r(\epsilon)$ as defined in the proof of Proposition 3.3 is in Λ.

Hence, for all $r\in\{1,\ldots,k\}$, for all $i\in I_r$,

$$\delta(\bar{b}_i,p)\subset L_r,$$

since otherwise

$$\delta(\bar{b}_i,p+\epsilon p_r)\cap L_r=\emptyset.$$

Note that, for every $r\in\{1,\ldots,k\}$, the economy $\mathscr{L}_{|I_r}\times L_r(\bar{b},\bar{\omega})$ has no proper self-sufficient subset. Thus, by Proposition 3.1(iii) of Ref. 7, for every $r\in\{1,\ldots,k\}$, the equilibrium price correspondence

$$P_{|r}:(\mathbb{R}_+^{L_r})^{I_r}\times(\mathbb{R}_+^{L_r})^{I_r}\rightarrow\mathbb{R}_{++}^{L_r}$$

of $\mathscr{L}_{|I_r\times L_r}(b,\omega)$ is upper semicontinuous at $(\bar{b},\bar{\omega})$ and, since it reduces to the half line $\{\lambda p_{|L_r}|\lambda>0\}$ at $(\bar{b},\bar{\omega})$, it is continuous at this point. Denote by $Q_r(b_{|I_r\times L_r},\omega_{|I_r\times L_r})$ its canonical injection into $\mathbb{R}^L$ intersected with the set

$$\{x\in\mathbb{R}^L|\|x\|=\|p_{|L_r}\|\}.$$

For $(b,\omega)\in\mathscr{M}(\bar{b},\bar{\omega})$, let

$$Q(b,\omega)=\sum_{r=1}^{k}Q_r(b_{|I_r\times L_r},\omega_{|I_r\times L_r}).$$

Note that this correspondence is continuous and single valued at $(\bar{b},\bar{\omega})$.

For every n, let $p^n\in Q(b^n,\omega^n)$. So, p^n converges to p. It is sufficient to prove that p^n is an equilibrium price of $\mathscr{L}(b^n,\omega^n)$ for all n large enough. For all $r\in\{1,\ldots,k\}$, p is an equilibrium price of $\mathscr{L}_{|I_r}(\bar{b},\bar{\omega})$, which is the restriction of $\mathscr{L}(\bar{b},\bar{\omega})$ to $I_r\times L$. For all n, for all r, $p^n_{|L_r}$ is an equilibrium price of $\mathscr{L}_{|I_r\times L_r}(b^n,\omega^n)$. Then, by the convergence of p^n to p and the fact that, for all $r\in\{1,\ldots,k\}$, for all $i\in I_r$, $\delta(\bar{b}_i,p)\subset L_r$, there exists for all $r\in\{1,\ldots,k\}$ some n_r such that, for all $n\geq n_r$, for all $i\in I_r$, $\delta(b_i^n,p^n)\subset L_r$. Thus, for all $n\geq n_r$, p^n is an equilibrium price of $\mathscr{L}_{|I_r}(b^n,\omega^n)$. Let

$$\bar{n}=\max_{r\in\{1,\ldots,k\}}n_r.$$

Hence, for all $n\geq\bar{n}$, p^n is an equilibrium price of the economy $\mathscr{L}(b^n,\omega^n)$. This leads to a contradiction. □

For $(\bar{b},\bar{\omega})\in\mathscr{W}$, let

$$\mathscr{M}'(\bar{b},\bar{\omega})=\{(b,\omega)\in\mathscr{W}|\forall i\in I,\ \operatorname{supp}\omega_i\subset\operatorname{supp}\bar{\omega}_i\}.$$

Corollary 4.1. For every $(\bar{b},\bar{\omega})\in\mathscr{W}$, the correspondence

$$P:\mathscr{M}'(\bar{b},\bar{\omega})\rightarrow\mathbb{R}_{++}^L$$

is lower semicontinuous.

Proof. For every $(\bar{b}, \bar{\omega}) \in \mathscr{W}$ and for every $(b, \omega) \in \mathscr{M}'(\bar{b}, \bar{\omega})$,

$$\mathscr{M}'(\bar{b}, \bar{\omega}) \subset \mathscr{M}(b, \omega).$$

By the previous proposition, for every $(b, \omega) \in \mathscr{M}'(\bar{b}, \bar{\omega})$, $P: \mathscr{M}(b, \omega) \to \mathbb{R}^L_{++}$ is lower semicontinuous at (b, ω) and hence also $P: \mathscr{M}'(\bar{b}, \bar{\omega}) \to \mathbb{R}^L_{++}$. □

Note that $P: \mathscr{W} \to \mathbb{R}^L_{++}$ is convex valued by Proposition 3.1(iii) of Ref. 7. Therefore, $P: \mathscr{M}'(\bar{b}, \bar{\omega}) \to \mathbb{R}^L_{++}$ admits continuous selections. This property may come in useful in a variety of applications. For example, in Ref. 10, it was crucial to study the approachability of hierarchic equilibria (Ref. 11) in convex economies, by dividend equilibria of economies with discrete consumption sets. Also by the above result, P is lower semicontinuous on

$$\mathscr{W}_\omega = \{b \in (\mathbb{R}^L_+)^I \mid (b, \omega) \in \mathscr{W}\}.$$

This could be exploited when one is interested only in changes of the utility function (for example, if one is interested in the impact of taxes on financial assets in the spirit of Ref. 12) or in strategic models as initiated in Ref. 13 (where agents may try to manipulate the market price by lying about their true utility function).

The correspondence $P: \mathscr{W} \to \mathbb{R}^L_{++}$ is not lower semicontinuous. In fact, for every $(b, \omega) \in \mathscr{W}$ and every $p \in P(b, \omega)$, there exists $t \in \mathbb{R}^{LI}_+$ such that, for all $\mu > 0$,

$$P(b, \omega + \mu t) = \{\lambda p \mid \lambda > 0\}.$$

Indeed, let $(I_r)_{r=1}^k$ and $(L_r)_{r=1}^k$ as in Section 3. For every $r \in \{1, \dots, k\}$, choose $i_r \in I_r$ and $h_r \in \delta(b_{i_r}, p)$. Let

$$t_{i_1 h_r} = p_{h_1}, \qquad \text{for } r \in \{2, \dots, k\},$$
$$t_{i_1 h} = 0, \qquad \text{otherwise.}$$

Let $t_{i_r h_1} = p_{h_r}$ for $r \in \{2, \dots, k\}$, let

$$t_{i_r h} = 0, \qquad \text{otherwise.}$$

For all other i, let $t_{ih} = 0$ for all $h \in L$. One checks easily that, for all $\mu > 0$,

$$P(b, \omega + \mu t) = \{\lambda p \mid \lambda > 0\}.$$

Therefore, the lower semicontinuity of $P: \mathscr{W} \to \mathbb{R}^L_{++}$ fails at all points (b, ω) where $P(b, \omega)$ is not a half line. It is straightforward to construct such examples.

5. Equilibrium Allocations

Once a point $(x, p) \in X(b, \omega) \times P(b, \omega)$ is known, one may use extensively the information that we gain from the knowledge of an equilibrium point for a characterization of the set $X(b, \omega)$. To use this characterization, one needs to know the set

$$G(b, p) = \{(i, h) \in I \times L | h \in \delta(b_i, p)\}.$$

This may be seen as a bipartite graph with vertices $I \cup L$ and an edge between vertices $(i, h) \in I \times L$ if and only if $(i, h) \in G(b, p)$.

For every cycle $c = (i_1, h_1, \ldots, h_n, i_1)$ of $G(b, p)$, let $t^c \in \mathbb{R}^{LI}$ with $t^c_{ih} = 0$ if the edge ih is not part of the cycle, otherwise $t^c_{i_r h_r} = 1/p_{h_r}$ and $t^c_{i_{r+1} h_r} = -1/p_{h_r}$ for $r = 1, \ldots, n$, $i_{n+1} = i_1$. Let $C(b, p)$ be the set of cycles of $G(b, p)$.

Proposition 5.1. Let $(b, \omega) \in \mathscr{W}$ and $(x, p) \in X(b, \omega) \times P(b, \omega)$. Then,

$$X(b, \omega) = \left\{ x + \sum_{c \in C(b, p)} \lambda^c t^c \,\middle|\, \lambda^c \in \mathbb{R} \right\} \cap \mathbb{R}^{LI}_+.$$

Proof. It is easy to check that the right-hand side is included in $X(b, \omega)$. The converse is a consequence of Lemma 4.1(i) of Ref. 7. Applying this result, given a point $x' \in X(b, \omega)$, one may find iteratively cycles $c^1, \ldots, c^k \in C(b, p)$ and weights $\mu^{c1}, \ldots, \mu^{ck}$ such that, for every $r \in \{1, \ldots, k\}$, $\mathrm{supp}(x + \sum_{\rho=1}^{r} \mu^{c\rho} t^{c\rho} - x')$ is a proper subset of $\mathrm{supp}(x + \sum_{\rho=1}^{r-1} \mu^{c\rho} t^{c\rho} - x')$, with $x + \sum_{\rho=1}^{r} \mu^{c\rho} t^{c\rho}$ in $X(b, \omega)$. Of course, in less than $\#(L \times I)$ steps, we have

$$x + \sum_{r=1}^{k} \mu^{cr} t^{cr} = x'. \qquad \square$$

It is interesting to note that typically there exists a small number of cycles, in $G(b, p)$ none. More precisely, one may deduce from Proposition 4.4 of Ref. 7 that the set of economies $(b, \omega) \in (\mathbb{R}^L_+)^{2I}$ such that no cycle in $G(b, p)$ for $p \in P(b, \omega)$ exists contains an open dense subset of $(\mathbb{R}^L_+)^{2I}$.

Let us denote by $Z(b, \omega)$ the elements of $X(b, \omega)$ with minimal support.

Corollary 5.1. Let $(b, \omega) \in \mathscr{W}$. Then, $X(b, \omega) = \mathrm{co}\, Z(b, \omega)$, $Z(b, \omega) = \{x^1, \ldots, x^k\}$, and $r \neq r'$ implies $\mathrm{supp}\, x^r \neq \mathrm{supp}\, x^{r'}$.

Proof. The inclusion $\mathrm{co}\, Z(b, \omega) \subset X(b, \omega)$ is a consequence of the convexity of $X(b, \omega)$; see Proposition 3.1(ii) of Ref. 7. By the previous proposition, if two points are in $X(b, \omega)$, then of course the line going through these points intersected with $\mathbb{R}_+^{LI}$ belongs to $X(b, \omega)$. Therefore, $r \neq r'$ implies supp $x^r \neq \text{supp}\, x^{r'}$ and the extremal points of $X(b, \omega)$ are its elements with minimal support. □

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