EXISTENCE AND OPTIMALITY OF OLIGOPOLY EQUILIBRIA IN LINEAR EXCHANGE ECONOMIES

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Abstract

We consider a linear exchange economy and its successive replicas. We study the notion of Cournot-Walras equilibrium in which the consumers use the quantities of commodities put on the market as strategic variables. We prove that, generically, if the number of replications is large enough but finite, the competitive behaviour is an oligopoly equilibrium. Then, under a mild condition, which may be interpreted in terms of market regulation and/or market activity, we show that any sequence of oligopoly equilibria of successive replica economies converges to the Walrasian outcome and furthermore that every oligopoly equilibrium of large, but finite, replica is Pareto optimal. Consequently, under the same assumptions on the fundamentals of the economy, one has an asymptotic result on the convergence of oligopoly equilibria to the Walras equilibrium together with a generic existence result for the Cournot-Walras equilibrium.

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1. Introduction

The incentives of price-taking behaviour are often discussed in the literature. An ever growing literature justifies this assumption by asymptotic results, which establish that strategic behaviour leads approximately to the competitive outcome (see, for example, Gabszewicz and Vial 1972, Postelwaite and Roberts 1976). This kind of result is interesting, provided the existence of a sequence of Nash equilibria can be proven under the same assumptions. So one needs to study two questions: firstly, does an oligopoly equilibrium converge to the competitive outcome when the number of agents increases and secondly, does an oligopoly equilibrium exist. It is of course of crucial importance that a positive answer to both questions can be given under the same assumptions.

There exist a large variety of approaches modelling strategic behaviour. The case of economies where the oligopolists are the producers has been studied by Gabszewicz and Vial (1972) Roberts, (1980), Mas-Colell (1983), Novshek and Sonnenschein (1983). Here we will be interested in a Cournot type model, more precisely we will study the case of exchange economies using the Cournot-Walras approach proposed by Codognato and Gabszewicz (1991) and Gabszewicz and Michel (1997). The strategic variables of the agents are the quantities of commodities they put on the market. In a second step a Walras equilibrium is played with respect to the announced endowments. Taking into account the effect of these quantities on the equilibrium price vector, each consumer tries to improve his utility level. Agents may thus buy back part of the declared endowment or simply not sell it.

Another Cournot type approach for exchange economies was initiated by Shapley (1976) and Shapley and Shubik (1977). This case has already been treated by Sahi and Yao (1989). In the Shapley-Shubik market game, agents strategies are market orders, i.e. bids of quantities of some commodity they are willing to sell against some other commodity. Then the price is established such that the market clears. So one may of course loose an entire bid for an arbitrary low price. This is not the case in the Cournot-Walras equilibrium since the final allocation of each agent is individually rational.

A more general approach has been initiated by Hurwicz (1972) and Postelwaite and Roberts (1976) where agents strategies are the announcement of preferences or of preferences and endowments. In a second step a Walras equilibrium is played with respect to the announced characteristics. Allowing for such sophisticated strategies, almost anything becomes an equilibrium unless one imposes restrictions on the type of preferences and endowments one may announce (Otani and Sicilian 1990). The idea behind this is roughly the following: take an individually rational, feasible allocation (x_i) such that there exists a price p, for which for every consumer the allocation x_i is worth as much as his initial endowment at price p. Then every agent may submit the demand function: if the price is p then I want to buy x_i and otherwise I keep my initial endowment. Imposing some continuity and consistency on the preferences one may submit, Otani and Sicilian (1990) prove that such allocations remain an oligopoly equilibrium.

Mathematically the Cournot-Walras model is of course equivalent to the approach of Hurwicz (1972) and Postelwaite and Roberts (1976) by making the appropriate restrictions on the permitted characteristics to be declared. But, contrary to the most general case, the set of equilibria is relatively small and under some reasonable assumptions, it is non-empty.

We will study the oligopoly equilibria in the replica of a linear exchange economy. Considering linear utility functions may have several justifications.

For instance, if we consider that the consumers exchange several times on the market only a small part of their initial endowments, we can assume that their preferences are represented by a short-term utility function, which is the first order linear differential of the long-term utility function. This is already done in Champsaur and Cornet (1990) and Bottazzi (1994) where during an exchange process, the infinitesimal trade at every moment is defined as the equilibrium of a tangent linear economy.

Linear utility functions appear also naturally when we consider a financial market where the traders can only put limit price orders on the market (See, Mertens 1996). Note also that linear exchange economies are considered by Bottazzi and de Meyer (1999) to study the effect of taxes on asset prices.

We will address the problem of existence of a sequence of oligopoly equilibria converging to a Walras equilibrium. Most importantly, we establish conditions consistent with the existence result under which the entire non-empty set of Nash equilibria converges to a Walras equilibrium when the number of agents increases. We emphasise the fact that the assumptions are stated in terms of the fundamentals of the economy.

Our first result is that, in large enough, but finite, replica economies, the price taking behaviour is an oligopoly equilibrium, provided the economy is regular, which is generically true. An important aspect of this result is that it obviously implies the generic existence of a sequence of oligopoly equilibria converging to a Walras equilibrium. We obtain this result by showing that the best response of each consumer is to put his initial endowment on the market. This is possible since in Bonnisseau, Florig and Jofré (2001a,b), it is shown that the equilibrium price vector is Lipschitz continuous with respect to the initial endowments. Thus, we can use the tools of nonsmooth optimization to conclude. Nevertheless, this does not mean that the consumers do not have any influence on the equilibrium price like in an economy with a non-atomic continuum of agents.

Later, we give a market activity condition, consistent with our existence result, under which every sequence of oligopoly equilibria converges to a Walras equilibrium as the number of agents increases. Moreover, we establish that oligopoly equilibria are Pareto optimal provided the economy is large enough. Nevertheless, this does not mean that the convergence takes always place in a finite number of steps. Our condition is satisfied, for example if some consumer's strategy set is bounded away from the boundary of the consumption set. So, if a policy maker wants to regulate this kind of markets, it is sufficient to check that only one agent (and his replica) put a positive quantity of each commodity on the market. This ensures Pareto optimality of the outcome on a non-competitive large but finite market.

A list of papers establishing an asymptotic result includes Postelwaite and Roberts (1976), Safra (1985), Otani and Sicilian (1990), Codognato and Gabszewicz (1991), Jackson (1992), Lahmandi-Ayed (2001), Jackson and Manelli (1997). Our convergence result of oligopoly equilibria to the competitive outcome is not a consequence of similar known results. Jackson and Manelli (1997) work with smooth preferences and they impose a regularity condition on the aggregate strategies which is of course quite restrictive. In the Cournot-Walras approach, it would imply that for any profile of declared initial endowments the resulting economy is not irregular. There it seems to be very hard to exclude this a priori. In Otani and Sicilian (1990), the demand functions must be smooth and this does not hold here. It may be worth to note that even in a differential setting simple strategies as in the Cournot-Walras approach results in non-smooth demands (Bonnisseau and Rivera 1997) and therefore Otani and Sicilian's (1990) asymptotic result does not encompass the Cournot-Walras approach.

Existence of a sequence of Nash equilibria converging to a Pareto optimum, when the number of agents increases, seems to have been treated, only by Safra (1985) and Gul and Postelwaite (1992). Gul and Postelwaite establish existence of some Nash equilibria in an asymmetric information environment, but a crucial assumption is that the strategy sets are finite. Safra works with standard preferences. However, in his model the payoff functions are not always well defined. Moreover, the initial endowment is supposed to be in the interior of the strategy set, in order to be able to apply the implicit function theorem. Hence, consumers may pretend to own more of some commodities than they actually do and hence short-sales must be possible. Finally, his notion of equilibrium does not take into account that the strategies in quantities leads to a modification of the demand functions since in his model consumers "forget" the part of their initial endowment they withheld. This ensures him that the demand function is still smooth. He is then able to establish existence of almost competitive Nash equilibria, when the number of strategic agents is large, by applying the implicit function theorem (see also Roberts 1980, Novsheck and Sonnenschein 1983). However, in the Cournot-Walras approach, withholding part of the initial endowment leads to non-smooth demands (Bonnisseau and Rivera 1997) which do not satisfy the strong regularity condition necessary to apply a Lipschitz version of the implicit function theorem.

Otani and Sicilian (1982, 1990) construct a non-empty set of Nash equilibria when the consumers' strategy is the announcement of a demand correspondence. If consumers are forced to submit smooth demands, then Nash equilibria converge to a Walras equilibrium as the economy grows. Their asymptotic result can however not be coupled with their existence result since the existence proof works precisely because the demands there are far from being smooth.

In the following section, we present the model and we recall several results about linear exchange economies. The results are stated in the next section together with several examples, which show that our assumptions are in some sense minimal. The proofs are given in Appendix.

2. The Model

We consider a linear exchange economy with a finite set $L = \{1, \ldots, \ell\}$ of commodities and a finite set $I = \{1, \ldots, m\}$ of consumers. The consumption set of the ith consumer ($i \in I$) is R^L_+ and his utility function $u_i : R^L_+ \to R$ is defined by $u_i(x_i) = b_i \cdot x_i$ for some given vector $b_i \in R^L_{++}$[1]. We denote by $\omega = (\omega_i)_{i \in I} \in (R^L_+)^I$, the vector of initial endowments, and by $\mathcal{L}(\omega)$, the competitive economy $(b_i, \omega_i)_{i \in I}$. We assume that

[1] It would be possible to work with vectors b_i on the boundary of the positive orthant, by imposing conditions ensuring that for any "strategy", there exists a Walras equilibrium in the resulting competitive economy.

$\sum_{i \in I} \omega_i \in R^L_{++}$, which means that each commodity is actually available in the economy. Note that we do not require a strong survival assumption on the initial endowments. This is particularly relevant here since oligopoly models implicitly suppose that some commodities are held only by a small number of consumers.

A price is an element of R^L_{++}. Contrary to the case of Cournot-Walras equilibria in an economy with production (Gabszewicz and Vial 1972), the normalization does not have any effect on the oligopoly equilibrium. In the following, we normalize the price vectors in the simplex of R^L_{++}, that is $\Sigma = \{p \in R^L_{++} \mid \sum_{h \in L} p_h = 1\}$.

To study the notion of oligopoly equilibrium in the sense of Gabszewicz and Michel (1997), we associate with each consumer i, his strategy set S_i, which is a subset of R^L_+, the space of endowments. We posit the following assumption on the strategy sets.

Assumption S. For all $i \in I$,
(i) $S_i \subset \{\sigma \in R^L \mid 0 \leq \sigma \leq \omega_i\}$ and $\omega_i \in S_i$;
(ii) for every $s_i \in S_i$, $\{\sigma \in R^L \mid s_i \leq \sigma \leq \omega_i\} \subset S_i$.

Assumption S(i) means that the strategy of an agent is a vector of commodities, which is less than his initial endowments, and an agent has always the possibility to choose his initial endowments as strategy. One may imagine that consumers might be required to show the commodities they declare to possess. Assumption S(ii) means that an agent can always increase his strategy as long as it remains below his initial endowments. We can now define an economy as a collection

$$\mathcal{E} = (b_i, \omega_i, S_i)_{i \in I}.$$

Each agent i chooses a strategy $s_i \in S_i$ which is the initial endowments he puts on the market. Thus, his demand with respect to the price vector $p \in R^L_{++}$ is the solution of the following maximization problem :

$$\begin{cases} \text{maximize } b_i \cdot (\omega_i - s_i + x) \\ p \cdot x \leq p \cdot s_i \\ x \geq 0 \end{cases}$$

The term $\omega_i - s_i$ in the objective function means that the consumer takes into account the part of his initial endowments he left at home. We remark that the linearity of the utility functions implies that the solution is actually the standard demand $d_i(p, p \cdot s_i)$ of the consumer with respect to the initial endowment s_i. Then, the exchange among the consumers takes place according to a Walras equilibrium of the economy $\mathcal{L}(s)$ where the utility functions are the same as those of the original economy but the initial endowments are the strategies of the consumers (s_i). By Gale (1976), there exists a non-empty set of equilibrium price vectors $P(s) \subset \Sigma$, which need not be unique, the utility levels of the consumers are however unique. So we do not have the problem of multiple equilibria as in the standard approach which makes it very hard to define the Cournot-Walras game correctly. Furthermore, a Walras equilibrium can be computed by a finite algorithm (See, Eaves (1976)). Thus, after the market stage of the game, utility levels of the consumers are :

$$V_i(s) = v_i(p(s), p(s) \cdot s_i) + b_i \cdot (\omega_i - s_i)$$

for some equilibrium price vector $p(s)$ of $\mathcal{L}(s)$ and where v_i is the indirect utility function, that is $v_i(p, w) = w \max\{\frac{b_{ih}}{p_h} \mid h \in L\}$.

An oligopoly equilibrium is a Nash equilibrium of the game where the players are the agents of the economy, the strategy sets are the sets S_i and the payoff functions are the mappings V_i.

Definition 2.1: An oligopoly equilibrium of the economy $\mathcal{E} = (b_i, \omega_i, S_i)_{i \in I}$ is a m-tuple of strategies $s \in \prod_{i \in I} S_i$ such that for all $i \in I$ and for all $\sigma_i \in S_i$,

$$V_i(s) \geq V_i(s_{-i}, \sigma_i)$$

where s_{-i} is the $(m-1)$-tuple of strategies $s_{i'}$, $i' \neq i$.

Of course we want to compare the outcome of a Cournot-Walras equilibrium with the one of a Walras equilibrium: so let s be an oligopoly equilibrium and let (p, ξ) a Walras equilibrium of the competitive economy $\mathcal{L}(s)$. Let $x = (x_1, \ldots, x_m)$ be defined by $x_i = \omega_i - s_i + \xi_i$. Then, x is an attainable allocation of the economy $\mathcal{E}$ and for all i, $V_i(s) = b_i \cdot x_i$ and $p \cdot x_i = p \cdot \omega_i$. Consequently, one obtains an attainable allocation and there exists a price vector such that, for each consumer, his allocation is in the budget set with respect to this price vector.

If the consumers are not strategic at all, that is $S_i = \{\omega_i\}$ for all i, then the unique oligopoly equilibrium is the Walras equilibrium. If autarky is a feasible strategy for all consumers, that is $0 \in S_i$ for all i, then this is an oligopoly equilibrium.

It should be noted that the present approach is quite different from the market game literature initiated by Shapley (1976) and Shapley and Shubik (1977). Indeed, the mechanism for the price formation at the market stage takes into account the preferences of the consumers in the Cournot-Walras approach. Thus, the relative prices cannot be larger than an upper bound given by the utility functions. This also means that a consumer may buy back a commodity that he has put on the market. This is not the case in the Shapley-Shubik model where the relative prices are computed according to the rate between the demand and the supply. The following example illustrates this remark. Let us consider an economy with three consumers and two goods.

Exemple 1. Let $b_1 = b_2 = (1, 2), b_3 = (2, 1)$, $\omega_1 = \omega_2 = (\alpha, 0), \omega_3 = (0, \beta)$ with $\alpha > 2\beta > 0$. Strategy sets are given by $S_i = \{s \in R^2_+ \mid 0 \leq s \leq \omega_i\}$ for all i. It is easy to check that the competitive behaviour and autarky are the only Cournot-Walras equilibria whereas in the Shapley-Shubik market game there is a unique equilibrium - autarky. This is of course due to the fact that in the Shapley-Shubik market game, one may loose the entire quantity put on the market for an arbitrary low price. Only the strategy to live in autarky ensures individual rationality of the obtained allocation. We do not have this phenomenon with the Cournot-Walras model since individual rationality of the obtained allocation is ensured for any strategy.

For some results of the next section, we need to know that the equilibrium price is unique for every strategy profile. This is false for example, if autarky is feasible for each consumer, which means that $0 \in S_i$ for all i, since, for this strategy profile, any price is an equilibrium price vector. One can impose a relatively strong condition, which says that the strategy of a given consumer is always strictly positive, that is $\bar{S}_i \subset R^L_{++}$. This is in particular true, if a non strategic consumer has strictly positive initial endowments. We now state a weaker assumption, which implies market activity for all goods and a unique price for all strategy profiles.

Assumption U. There exists a finite family $\{i_1, ..., i_n\} \subset I$ and some $\varepsilon > 0$ such that :

(i) for all $h \in L$, there exists $\nu \in \{1, \ldots, n\}$ such that $s_{i_\nu h} > \varepsilon$ for all $s_{i_\nu} \in S_{i_\nu}$;

(ii) for all $\nu \in \{1, \ldots, n-1\}$, there exists $h \in L$ such that $s_{i_\nu h} > \varepsilon$ for all $s_{i_\nu} \in S_{i_\nu}$ and $s_{i_{\nu+1} h} > \varepsilon$ for all $s_{i_{\nu+1}} \in S_{i_{\nu+1}}$.

This type of assumption is standard in the literature (see e.g. Safra 1985, Lahmandi-

Ayed 2001).[2] It excludes trivial equilibria.

This Assumption means that for all strategy profiles, for all commodities h, at least one consumer of the family $\{i_1, ..., i_n\}$ puts a quantity of h greater than ε on the market and the strategies are connected in the sense that a pair of successive consumers puts a quantity greater than ε of the same commodity on the market.

One could interpret this assumption in various ways. Firstly, one could think of some regulation of the markets. Participating at the market implies that one has to offer at least a fixed $\varepsilon > 0$ out of some commodities. The assumption holds true in particular if the closure of one strategy set is included in R^L_{++}. So a policy maker could regulate only one agent (and his replica) in order to ensure the validity of the subsequent propositions.

Secondly, one could think of the economy consisting of big agents with strategy sets $\{s_i \in R^L_+ \mid 0 \leq s_i \leq \omega_i\}$ and a collection $\{i_1, ..., i_n\} \subset I$ representing n types of consumers, each type consisting of a continuum of negligible consumers. For each $i \in \{i_1, ..., i_n\}$, all consumers within this type have the same preferences b_i and their initial endowments add up to ω_i. For these consumers, the competitive behaviour is of course always a best reply and it would be sufficient that their endowments satisfy the above connectedness condition. Modelling these consumers explicitly as non-atomic sets of agents would require virtually no changes apart notations.

Finally, one could think that agents behave strategically on a subset of the set of goods (cf. Gabszewicz and Michel 1997). For example, they are strategic on markets of goods where they are big players and competitive on the market of goods where their size relative to other agents is very small.

We denote by $\mathcal{U} \subset (R^L_+)^I$ the endowments, for which there exists a unique equilibrium price vector, that is $\omega' \in \mathcal{U}$ if the economy $\mathcal{L}(\omega') = (b_i, \omega'_i)_{i \in I}$ has a unique normalized price vector. From Gale (1976), one deduces the following result:

Proposition 2.1. *Under Assumption U, $\prod_{i \in I} \bar{\text{co}} S_i$ is included in the interior (relative to $(R^L_+)^I$) of $\mathcal{U}$.*

We now recall the notion of regular initial endowments for linear exchange economies, taken from Bonnisseau, Florig and Jofré (2001b). If the initial endowments are regular,

[2] Safra (1985) assumes $S_i = \{s_i \in R^L_{++} \mid s_i \leq \bar{\omega}_i\}$ for some $\bar{\omega}_i >> \omega_i$, but then for his convergence result he considers an sequence of oligopoly equilibria, such that market activity is ensured at the limit. For this to hold, one needs an assumption of the type $\bar{S}_i \subset R^L_{++}$.

then the normalized equilibrium price vector is unique and smooth in a neighborhood.

Definition 2.2. The initial endowment $\omega \in (R^L_+)^I$ is regular, if the Walras equilibrium price p is unique (in Σ) and if for some Walras equilibrium allocation $(x_1, ..., x_m)$,

$$\text{for all } i = 1, ..., m, \text{ for all } h \in \delta_i(p), \ x_{ih} > 0,$$

where $\delta_i(p) = \{h \in L \mid \frac{b_{ih}}{p_h} = \max\{\frac{b_{ih'}}{p_{h'}} \mid h' \in L\}\}$.

The commodities in $\delta_i(p)$ are those that the ith consumer wants buy because the marginal rate of substitution with respect to any other commodity is greater or equal to the relative price. Thus, an economy is regular if an equilibrium allocation allows each consumer to obtain a positive amount of each commodity he wants to buy. We denote by Ω the set of regular initial endowments. By Bonnisseau, Florig and Jofré (2001a,b) and Bonnisseau and Florig (2002), the set Ω is open (in $(R^L_+)^I$), dense and of full Lebesgue measure.

We will consider replicas of the original economy to model the idea that the number of consumers becomes large and the relative weight of each agent becomes small. For every integer $k \geq 1$, we define the economy $\mathcal{E}^k$ as the k-th replica of the basic economy $\mathcal{E}$. In the economy $\mathcal{E}^k$, there are mk agents shared in m groups of k identical agents. Each agent of the ith group has the same characteristics b_i, ω_i and S_i. Similarly, we note $\mathcal{L}^k(\omega)$, the kth replication of the competitive economy $\mathcal{L}(\omega)$. In the following, if y is a vector of $(R^L_+)^{mk}$, we denote by $\bar{y}$ the average vector of $(R^L_+)^I$ defined by $\bar{y}_i = \frac{1}{k}\sum_{j=1}^k y_{ij}$.

3. Existence, Optimality and Asymptotic Behavior of Oligopoly Equilibria

Our first proposition establishes the generic existence of oligopoly equilibria for large, but finite, replica economies.

Proposition 3.1. *Let $\mathcal{E} = (b_i, \omega_i, S_i)_{i \in I}$ be an economy satisfying Assumption S(i). If $\omega \in (R^L_+)^I$ is a regular initial endowment, then there exists an integer k_0 such that for every $k \geq k_0$ the strategy (s^k_{ij}) defined by $s^k_{ij} = \omega_i$ for all (i, j) in $I \times \{1, ..., k\}$, is an oligopoly equilibrium of the economy $\mathcal{E}^k$.*

This proposition shows that, if the economy is finite, but sufficiently large and, if the initial endowment is regular, then price taking behaviour is an oligopoly equilibrium.

Regularity is a weak assumption, since almost all initial endowments are regular. Nevertheless, it does not mean that in a large finite economy, the agents have no power on the equilibrium prices. Indeed, other oligopoly equilibria may exist as the following examples show.

Example 2. Let us consider a two good-two consumer economy with $b_1 = (1, 0)$, $b_2 = (0, 1)$, $\omega_1 = \omega_2 = (1, 1)$ and $S_i = \{s \in R^2_+ \mid 0 \leq s \leq \omega_i\}$. Then, without any replication, the competitive behaviour is an oligopoly equilibrium among an infinity of other equilibria. In particular, autarky is not the only oligopoly equilibrium. Indeed, if say agent 1 puts a quantity $\lambda > 0$ of good two on the market, then he obtains the entire quantity μ of good one that agent 2 put on the market and vice versa. So the payoff depends not on his own strategy, but on the strategy of the other consumer as long as market activity is ensured. One easily checks that the set of allocations which may be supported by an oligopoly equilibrium is:

$$\{x \in (R^2_+)^2 | x_1 = (1 + \mu, 1 - \lambda), x_2 = (1 - \mu, 1 + \lambda), (\lambda, \mu) \in]0, 1]^2\} \cup \{(\omega_1, \omega_2)\}.$$

For the k times replicated economy with $k \geq 2$ only two outcomes are supported by oligopoly equilibria - autarky and the competitive outcome.

Example 3. In the following economy, we obtain a sequence of Cournot-Walras equilibria which are neither the competitive equilibrium nor the autarky equilibrium. Let us consider the competitive economy $\mathcal{L}(\omega)$ with two agents and two commodities defined by $b_1 = (1, 1), b_2 = (1, 2)$, $\omega_1 = (1, 2)$ and $\omega_2 = (1, 1)$. The unique Walras equilibrium of this economy is $p = (1, 1)$, $x_1 = (2, 1)$, $x_2 = (0, 2)$, thus the initial endowments are regular. We draw attention to the fact that in this case consumer 1 does not gain anything from going to the market. Let $S_1 = \{s \in R^2_+ \mid s_1 = 1,\ 0 \leq s_2 \leq 2\}$, $S_2 = \{\omega_2\}$, thus only trader 1 can behave strategically. Our previous result shows that the competitive outcome is an oligopoly equilibrium if the economy is sufficiently replicated. We now exhibit a sequence of oligopoly equilibria, for which the outcome is different from the Walrasian one for every finite k, and only at the limit, the Walrasian equilibrium price is attained.

It is easy to check that, for $k = 1$, $s_1^1 = (1, 1/2)$, $s_2^1 = \omega_2$ is an oligopoly equilibrium. The associated price-allocation couple is, $p^1 = (1, 2)$, $x_1^1 = (2, 3/2)$, $x_2^1 = (0, 3/2)$. For $k \geq 2$ and every $j \in \{1, 2, ..., k\}$, $s_{1j}^k = (1, 1 - 1/k)$, $s_{2j}^k = \omega_2$ is an oligopoly equilibrium, and $p^k = (1 - 1/k, 1)$, $x^k_{1j} = (2, 1 + 1/k)$, $x^k_{2j} = (0, 2 - 1/k)$ is the associated price-allocation couple. Note that the limit of $(\bar{s}^k_1, \bar{s}^k_2)$ is equal to $((1,1),(1,1))$ which is not a regular initial endowment with respect to the utility vectors b_1 and b_2. It is worth noting that one may perturb the parameters of the example without changing the nature of the result.

Note that the allocations (x^k) are Pareto optimal. The surprising fact is that this is not specific to the above example as the following proposition shows.

Proposition 3.2. *Let $\mathcal{E} = (b_i, \omega_i, S_i)_{i \in I}$ be an economy satisfying Assumptions S and U. For every positive integer k, let s^k be an oligopoly equilibrium of the economy $\mathcal{E}^k$ and let p^k be a corresponding equilibrium price vector. Then, there exists an integer k_0 such that for every $k \geq k_0$, all allocations associated with the oligopoly equilibrium s^k are Pareto optimal.*

So far, we did not exclude the existence of sequences of oligopoly equilibria not converging to the competitive outcome. Such undesirable sequences exist as shows the following example.

Example 4. Consider a regular competitive economy $\mathcal{L}(\omega)$ with two agents and two commodities. Let $b_1 = (2,1), b_2 = (1,2)$, $\omega_1 = (0,1)$ and $\omega_2 = (1,0)$. The unique Walras equilibrium of this economy is $p = (1,1)$, $x_1 = (1,0)$, $x_2 = (0,1)$. Let $S_i = \{s \in R^2_+ \mid 0 \leq s \leq \omega_i\}$. For $k = 1$, the only oligopoly equilibrium is the no-trade equilibrium $s_1 = s_2 = (0,0)$ and the associated allocation and prices are $x_i = \omega_i$, $i = 1,2$ and p for all $p \in R^2_{++}$. The replication of the economy does not eliminate the no-trade equilibrium. For $k = 2$, the oligopoly equilibria are $s_{11} = s_{12} = (0,t)$ and $s_{21} = s_{22} = (t,0)$ for any $t \in [0, 1/2]$. The associated allocations are $x_{1j} = (t, 1-t)$, $j = 1,2$, $x_{2j} = (1-t, t)$, $j = 1,2$ and for $t > 0$ the associated price is $p = (1,1)$. For every $k \geq 3$, there are exactly two oligopoly equilibria, the no-trade equilibrium and the competitive behaviour $s_{ij} = \omega_i$ for $i = 1,2$ and $j = 1, \ldots, k$.

The next result excludes sequences of equilibria not converging to the competitive outcome. It plays also an important role in the proof of the previous propositions.

Proposition 3.3. *Let $\mathcal{E} = (b_i, \omega_i, S_i)_{i \in I}$ be an economy satisfying Assumptions S(i) and U. For every positive integer k, let s^k be an oligopoly equilibrium of the economy $\mathcal{E}^k$ and* *let (p^k, x^k) be a price vector and a feasible allocation associated with this equilibrium. Then, the sequence (p^k) converges to $p(\omega)$. Furthermore, if (j_k) is a sequence such that for all k, $j_k \in \{1, \ldots, k\}$, then $(V^k_{ij_k}(s^k))$ converges to $v_i(p(\omega), p(\omega) \cdot \omega_i)$. Finally, every cluster point of the sequence of average consumption plans $(\bar{x}^k)$ is an equilibrium allocation of $\mathcal{L}(\omega)$.*

If all consumers have finite strategy sets, then Proposition 3.3 implies that for all k large enough, all oligopoly equilibria are equivalent to the Walrasian outcome.

There are several results in the literature similar to Proposition 3.3. Nevertheless, as already stressed in the introduction, the proposition cannot be deduced from them. Note that the strategies of the agents may not converge to ω_i, since it is possible that other strategies lead to the competitive outcome. As for the equilibrium allocations, we cannot have a more precise result since they are not unique at equilibrium.

Without Assumption U, the asymptotic result does not hold. For example, if $0 \in \prod_{i \in I} S_i$ and if all consumers $i \in I$ declare $s_i = 0$, then this is a trivial oligopoly equilibrium and it is of course not eliminated by replication. Note that every price vector is an equilibrium price vector if each consumer chooses the strategy 0. If the equilibrium price is not unique, then by Bonnisseau, Florig and Jofré (2001a), there exists a nonempty proper subset $A \subset I$, such that agents in A do not exchange anything with the others, $B = I \setminus A$. Sequences of oligopoly equilibria not converging to the Walrasian outcome correspond therefore to a sort of generalization of the trivial oligopoly equilibrium. The market is split into several groups and no exchanges across these groups take place.

4. Appendix

We first recall that the equilibrium price p is locally Lipschtiz continuous on the interior of $\mathcal{U}$ with respect to the initial endowments. This is a consequence of Proposition 2.1 (iv) in Bonnisseau, Florig and Jofré (2001a) and Bonnisseau-Florig (2002). Under Assumption U, since $\prod_{i \in I} \bar{\text{co}} S_i \subset \text{int}\mathcal{U}$, one deduces that the mapping p is locally Lipschitz continuous on $\prod_{i \in I} \bar{\text{co}} S_i$. In the following, we also denote by p the extension of p to $(R^L)^I$ defined by $p(\omega) = p(\pi(\omega))$ where π is the projection on $\prod_{i \in I} \bar{\text{co}} S_i$. This mapping is locally Lipschitz continuous on the whole space. Finally, one remarks that if $s^k \in \prod_{i \in I} (S_i)^k$ is an oligopoly equilibrium of the economy $\mathcal{E}^k$ and (p^k, x^k) are a price vector and an attainable allocation associated with this equilibrium, then the structure of the demand correspondence with linear utility functions implies that p^k

is a Walrasian equilibrium price vector of $\mathcal{L}(\bar{s}^k)$ where $\bar{s}_i^k = \frac{1}{k}\sum_{j=1}^k s_{ij}^k$ for all i and $\bar{\xi}_i^k = \frac{1}{k}\sum_{j=1}^k (x_{ij}^k + s_{ij}^k - \omega_i)$ for all i is an equilibrium allocation of $\mathcal{L}(\bar{s}^k)$.

Proof of Proposition 3.3. Let $s^k \in \prod_{i\in I}(S_i)^k$ be an oligopoly equilibrium of the economy $\mathcal{E}^k$ and let (p^k, x^k) be a price vector and an attainable allocation associated with this equilibrium. Let (i,j) be an element of $I \times \{1,\ldots,k\}$, then we note s_{-ij}^k the $(km-1)$-tuple of strategies $s_{i'j'}^k$ with $(i',j') \neq (i,j)$. We note $\sigma^k = (s_{-ij}^k, \sigma_i)$ the element of $\prod_{i\in I}(S_i)^k$ such that $\sigma_{i'j'}^k = s_{i'j'}^k$ for all $(i',j') \neq (i,j)$ and $\sigma_{ij}^k = \sigma_i$. In other words, σ^k is the mk-tuple of strategies derived from s^k by replacing the strategy s_{ij}^k of the agent ij by $\sigma_i \in S_i$.

For all (i,j), let $\sigma^k = (s_{-ij}^k, \omega_i)$. One has :

$$v_i(p(\sigma^k), p(\sigma^k)\cdot\omega_i) \leq V_{ij}^k(s^k) = b_i \cdot x_{ij}^k \leq v_i(p^k, p^k\cdot\omega_i)$$

The first inequality comes from the fact that s^k is an oligopoly equilibrium and the second one from the fact that x_{ij}^k belongs to the budget set defined by p^k and $p^k \cdot \omega_i$.

The remainder of the proof is divided into three steps.

Step 1. The sequence (p^k) converges to $p(\omega)$ the Walrasian equilibrium price vector of $\mathcal{L}(\omega)$.

Proof: The function $v_i(p(s), p(s)\cdot\omega_i)$ is continuous on $\prod_{i\in I}\overline{\text{co}}S_i$, hence, uniformly continuous. For all $\varepsilon > 0$, there exists $\eta_\varepsilon > 0$ such that for all $(s,s') \in (\prod_{i\in I}\overline{\text{co}}S_i)^2$, $\sum_{i\in I}\|s_i - s_i'\| < \eta_\varepsilon$ implies $|v_i(p(s), p(s)\cdot\omega_i) - v_i(p(s'), p(s')\cdot\omega_i)| < \varepsilon$ for all i.

Since the prices remain in a compact set, it suffices to prove that every converging subsequence of (p^k) (again denoted (p^k)) converges to $p(\omega)$. We can assume without any loss of generality that $(\bar{x}^k)$ converges to $\tilde{x}$, an attainable allocation of $\mathcal{L}(\omega)$.

Let $\varepsilon > 0$ and let k_ε large enough so that $\frac{1}{k_\varepsilon}\max_{i\in I}\{\|\omega_i\|\} < \eta_\varepsilon$. Let $\sigma^k = (s_{-ij}^k, \omega_i)$. For all $k \geq k_\varepsilon$, $\sum_{i'\in I}\|\bar{s}_{i'}^k - \bar{\sigma}_{i'}^k\| = \frac{1}{k}\|\omega_i - s_{ij}^k\| \leq \frac{1}{k}\|\omega_i\| < \eta_\varepsilon$. Consequently, from the definition of η_ε and the fact that $p^k = p(\bar{s}^k)$ and $p^k(\sigma^k) = p(\bar{s}_{-i}^k, \bar{s}_i^k + \frac{1}{k}(\omega_i - s_{ij}^k))$, one has for all $i \in I$

$$|v_i(p(s_{-ij}^k, \omega_i), p(s_{-ij}^k, \omega_i)\cdot\omega_i) - v_i(p^k, p^k\cdot\omega_i)| < \varepsilon$$

Thus, for all $k \geq k_\varepsilon$, $b_i \cdot x_{ij}^k \in [v_i(p^k, p^k\cdot\omega_i) - \varepsilon, v_i(p^k, p^k\cdot\omega_i)]$. Hence $b_i \cdot \bar{x}_i^k = \frac{1}{k}\sum_{j=1}^k b_i \cdot x_{ij}^k \in [v_i(p^k, p^k\cdot\omega_i) - \varepsilon, v_i(p^k, p^k\cdot\omega_i)]$ for all i. Since $(\bar{x}^k)$ converges to $\tilde{x}$, we obtain at the limit, $b_i \cdot \tilde{x}_i \in [v_i(\tilde{p}, \tilde{p}\cdot\omega_i) - \varepsilon, v_i(\tilde{p}, \tilde{p}\cdot\omega_i)]$ for all i where $\tilde{p}$ is the limit of the converging subsequence (p^k). Since the above inclusion is true for all $\varepsilon > 0$, one deduces that $b_i \cdot \tilde{x}_i = v_i(\tilde{p}, \tilde{p} \cdot \omega_i)$ for all i. Together with the fact that $\tilde{x}$ is an attainable allocation of $\mathcal{L}(\omega)$, these equalities imply that $(\tilde{p}, \tilde{x})$ is a Walras equilibrium of $\mathcal{L}(\omega)$. The uniqueness of the equilibrium price vector of $\mathcal{L}(\omega)$ implies $\tilde{p} = p(\omega)$ which ends the proof of this step. ■

Step 2. $(V^k_{ij_k}(s^k))$ converges to $v_i(p(\omega), p(\omega) \cdot \omega_i)$ and for every cluster point $\tilde{x}$ of the sequence $(\bar{x}^k)$ is an equilibrium allocation of $\mathcal{L}(\omega)$.

Proof: Using the notation and the argument of the previous step, for all $\varepsilon > 0$, for all $k \geq k_\varepsilon$ one has $V^k_{ij_k}(s^k) \in [v_i(p^k, p^k \cdot \omega_i) - \varepsilon, v_i(p^k, p^k \cdot \omega_i)]$, which leads to the result since (p^k) converges to $p(\omega)$. This ends the proof of Proposition 3.3. ■

Proof of Proposition 3.2. We recall the notation for $p \in R^L_{++}$,

$$\delta_i(p) = \{h \in L \mid \frac{b_{ih}}{p_h} = \max\{\frac{b_{ih'}}{p_{h'}} \mid h' \in L\}\}$$

Step 1. There exists an integer k_0 such that for every $k \geq k_0$, for every $(i,j) \in I \times \{1, \ldots, k\}$ and every $h \notin \delta_i(p(\omega))$, $s^k_{ijh} = \omega_{ih}$.

Proof: We denote by $C(\bar{s}^k)$ the set of cluster points of the sequence $(\bar{s}^k)$. Clearly $C(\bar{s}^k) \subset \prod_{i \in I} \bar{\text{co}} S_i$. Let $\bar{s} \in C(\bar{s}^k)$. By Proposition 3.3 and the continuity of the mapping p , $p(\bar{s}) = \lim_{k\to\infty} p^k = p(\omega)$ where $p(\omega)$ denotes the unique equilibrium price vector of the economy $\mathcal{L}(\omega)$. Thus $\delta_i(p(\omega)) = \delta_i(p(\bar{s}))$ for all $i \in I$.

We now prove the claim for a consumer i such that there exists $\bar{s} \in C(\bar{s}^k)$ with $\bar{s}_i = 0$. We consider a subsequence of $(\bar{s}^k)$, again denoted $(\bar{s}^k)$, such that $(\bar{s}^k_i)$ converges to $\bar{s}_i$. By Proposition 3.3, one has $v_i(p(\omega), p(\omega) \cdot \omega_i) = \lim_{k\to\infty} \frac{1}{k} \sum_{j=1}^k V^k_{ij}(s^k) = \lim_{k\to\infty} p^k \cdot \bar{s}^k_i \max_h\{\frac{b_{ih}}{p^k_h}\} + b_i \cdot (\omega_i - \bar{s}^k_i) = b_i \cdot \omega_i$. This implies that ω_i belongs to the demand for the price $p(\omega)$. Consequently, the support of ω_i is included in $\delta_i(p(\omega))$ which implies that for all $s_i \in S_i$, the support of s_i is included in $\delta_i(p(\omega))$. Thus for all $h \notin \delta_i(p(\omega))$, for all integers k, $s^k_{ijh} = \omega_{ih} = 0$.

We now consider the consumers in $I^+ = \{i \in I \mid \forall \bar{s} \in C(\bar{s}^k), \bar{s}_i \neq 0\}$. For all $\bar{s} \in C(\bar{s}^k)$, for all $i \in I^+$ there exists a commodity $h^{\bar{s}}_i \in L$ and ξ, an equilibrium allocation of $\mathcal{L}(\bar{s})$, such that $\xi_{ih^{\bar{s}}_i} > 0$.

By the continuity of the price function in a neighbourhood of $\bar{s}$ and by the lower semi continuity of the Walras equilibrium allocation correspondence (Proposition 5.1 (i) in Bonnisseau, Florig and Jofré 2001a), there exists $r^{\bar{s}} > 0$ such that $\bar{B}(\bar{s}, r^{\bar{s}}) \subset \mathcal{U}$ and for all i,

$h_i^{\bar{s}} \in \delta_i(p(\omega')) \subset \delta_i(p(\bar{s}))$ for all $\omega' \subset \bar{B}(\bar{s}, r^{\bar{s}})$. $(*)$

Since $C(\bar{s}^k)$ is compact, there exists a finite family $(\bar{s}^1, \ldots, \bar{s}^n)$ in $C(\bar{s}^k)$ such that $C(\bar{s}^k) \subset \cup_{\nu=1}^n B(\bar{s}^\nu, \frac{r^{\bar{s}^\nu}}{2})$. From now on, we write r^ν and h_i^ν instead of $r^{\bar{s}^\nu}$ and $h_i^{\bar{s}^\nu}$. Let $\rho = \min\{\frac{r^\nu}{2} \mid \nu = 1, \ldots, n\}$.

Let k_1 be an integer large enough such that for all $k \geq k_1$, $\bar{s}^k \in \cup_{\nu=1}^n B(\bar{s}^\nu, \frac{r^\nu}{2})$ and $\frac{1}{k}\|\omega_i\| < \rho$ for all $i \in I^+$. Since for all $k \geq k_1$ $\bar{s}^k \in \cup_{\nu=1}^n B(\bar{s}^\nu, \frac{r^\nu}{2})$, for all (i,j), for all $\sigma_i \in S_i$, (s_{-ij}^k, σ_i) belongs to $B(\bar{s}^\nu, r^\nu)$, for some ν and $h_i^\nu \in \delta_i(p(s_{-ij}^k, \sigma_i))$. Thus,

$$V_{ij}^k(s_{-ij}^k, \sigma_i) = p(\bar{s}_{-i}^k, \bar{s}_i^k + \frac{1}{k}(\sigma_i - s_{ij}^k)) \cdot \sigma_i \frac{b_{ih_i^\nu}}{p_{h_i^\nu}(\bar{s}_{-i}^k, \bar{s}_i^k + \frac{1}{k}(\sigma_i - s_{ij}^k))} + b_i \cdot (\omega_i - \sigma_i).$$

In the following, if f is a locally Lipschitz continuous mapping, we denote by ∂f its generalized gradient in the sense of Clarke (1983). Let $\bar{h} \notin \delta_i(p(\omega))$. Since s_{ij}^k is the best response of agent ij, if $s_{ij\bar{h}}^k < \omega_{i\bar{h}}$ for some commodity $\bar{h}$, Assumption S implies that the consumer may increase the quantity of good $\bar{h}$ he puts on the market. Thus, the first order necessary conditions imply that there exists $y \in \partial V_{ij}^k(s^k)$ such that $y_{ij\bar{h}} \leq 0$. From Clarke (1983), there exists $(\tilde{u}^h) \in \prod_{h \in L} \partial p_h(\bar{s}^k)$ and $\tilde{w}^{h_i^\nu} \in \partial p_{h_i^\nu}(\bar{s}^k)$ such that

$$\begin{aligned} y_{ij} &= \Big(\sum_{h \in L} \frac{1}{k} s_{ijh}^k \tilde{u}^h + p(\bar{s}^k)\Big) \frac{b_{ih_i^\nu}}{p_{h_i^\nu}(\bar{s}^k)} - p(\bar{s}^k) \cdot s_{ij}^k \frac{b_{ih_i^\nu}}{(p_{h_i^\nu}(\bar{s}^k))^2} \frac{1}{k} \tilde{w}^{h_i^\nu} - b_i \\ &= \frac{b_{ih_i^\nu}}{k p_{h_i^\nu}(\bar{s}^k)} \Big(\sum_{h \in L} s_{ijh}^k \tilde{u}^h - \frac{p(\bar{s}^k) \cdot s_{ij}^k}{p_{h_i^\nu}(\bar{s}^k)} \tilde{w}^{h_i^\nu}\Big) + p(\bar{s}^k) \frac{b_{ih_i^\nu}}{p_{h_i^\nu}(\bar{s}^k)} - b_i \end{aligned}$$

To study the sign of this expression, we first state some remarks. From condition (*), for all ν, $\omega' \in \bar{B}(\bar{s}^\nu, r^\nu)$, for all $i \in I^+$ and all $h_0 \notin \delta_i(p(\omega)) = \delta_i(p(\bar{s}^\nu))$,

$$0 < p_{h_0}(\omega') \frac{b_{ih_i^\nu}}{p_{h_i^\nu}(\omega')} - b_{ih_0}$$

Hence there exists $\eta > 0$, such that , for all ν, $\omega' \in \bar{B}(\bar{s}^\nu, r^\nu)$, $i \in I^+$ and $h_0 \notin \delta_i(p(\omega))$,

$$\eta \leq p_{h_0}(\omega') \frac{b_{ih_i^\nu}}{p_{h_i^\nu}(\omega')} - b_{ih_0}$$

Since the generalized gradient is an upper semi-continuous correspondence with compact values, there exists a real number $M > 0$ such that for all ν, $h_0 \in L$, $i \in I^+$, $\sigma_i \in S_i$, $\omega' \in \bar{B}(\bar{s}^\nu, r^\nu)$, $(u^h) \in \prod_{h \in L} \partial p_h(\omega')$ and $w^{h_i^\nu} \in \partial p_{h_i^\nu}(\omega')$

$$\frac{b_{ih_i^\nu}}{p_{h_i^\nu}(\omega')}\Big(\sum_{h\in L}\sigma_{ih}u^h_{h_0}-\frac{p(\omega')\cdot\sigma_i}{p_{h_i^\nu}(\omega')}w^{h_i^\nu}_{h_0}\Big)>-M$$

We now choose an integer k_0 such that $k_0\geq k_1$ and $k_0>\frac{M}{\eta}$. Let $k\geq k_0$. Since $\bar h\notin\delta_i(p(\omega))=\delta_i(p(\bar s^\nu))$ for all ν, for all $i\in I^+$, for all $y\in\partial V^k_{ij}(s^k)$, one has $y_{ijh}\geq -\frac{1}{k}M+\eta>0$. The last inequality comes from the fact that $k\geq k_0>\frac{M}{\eta}$. Consequently, from the first order necessary condition, one deduces that $s^k_{ij\bar h}=\omega_{i\bar h}$. This ends the proof of the first step. ∎

Step 2. For $k\geq k_0$, every allocation x^k associated with the oligopoly equilibrium s^k is Pareto optimal.

Proof: For $k\geq k_0$ and every ij in I^k, $x^k_{ij}=\omega_i-s^k_{ij}+\xi^k_{ij}$ with $\xi^k_{ij}\in d_i(p^k,p^k\cdot s^k_{ij})$. Since $k\geq k_0\geq k_1$, $\bar s^k\in B(\bar s^\nu,\frac{r^\nu}{2})$ for some ν, which implies that $\delta_i(p(\bar s^k))\subset\delta_i(p(\bar s^\nu))=\delta_i(p(\omega))$. Consequently, by Step 1, for every $h\notin\delta_i(p(\omega))$, $x^k_{ijh}=0$. Hence $p(\omega)$ supports the allocation and thus, x^k is a Pareto optimum of the economy $\mathcal{E}^k$. ∎

Proof of Proposition 3.1. From Bonnisseau, Florig and Jofré (2001a,b) and Bonnisseau and Florig (2002), since ω is a regular initial endowment, there exists $r>0$ such that for all $\omega'\in(R^L_+)^I\cap\bar B(\omega,r)$, the equilibrium price vector is unique and for all $i\in I$, $\delta_i(p(\omega'))=\delta_i(p(\omega))$.

Let $\hat s$ be the strategy defined by $\hat s_{ij}=\omega_i$ for all i,j. For k large enough, for every $\sigma_i\in\bar{\text{co}}S_i$, the average strategy $(\omega_{-i},\omega_i+\frac{1}{k}(\sigma_i-\omega_i))\in\bar B(\omega,r)$. Using the same argument as in the first step of the proof of Proposition 3.2, one concludes that for k large enough, the best response $\bar\sigma_i$ of agent i,j to $\hat s_{ij}$ in $\bar{\text{co}}S_i$ satisfies $\bar\sigma_{ih}=\omega_{ih}$ for all $h\notin\delta_i(p(\omega))$.

In a linear exchange economy, the equilibrium price vector does not change, if one increases the initial endowment of the i-th consumer for commodities in $\delta_i(p)$, where p is the equilibrium price vector. Thus, one deduces that $p(\omega_{-i},\omega_i+\frac{1}{k}(\sigma_i-\omega_i))=p(\omega)$. This implies that ω_i is also a best response of agent i,j to $\hat s_{ij}$ in $\bar{\text{co}}S_i$. So, since $S_i\subset\bar{\text{co}}S_i$, ω_i is also a best response of agent i,j to $\hat s_{ij}$ in S_i, which shows that $\hat s$ is an oligopoly equilibrium. ∎

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